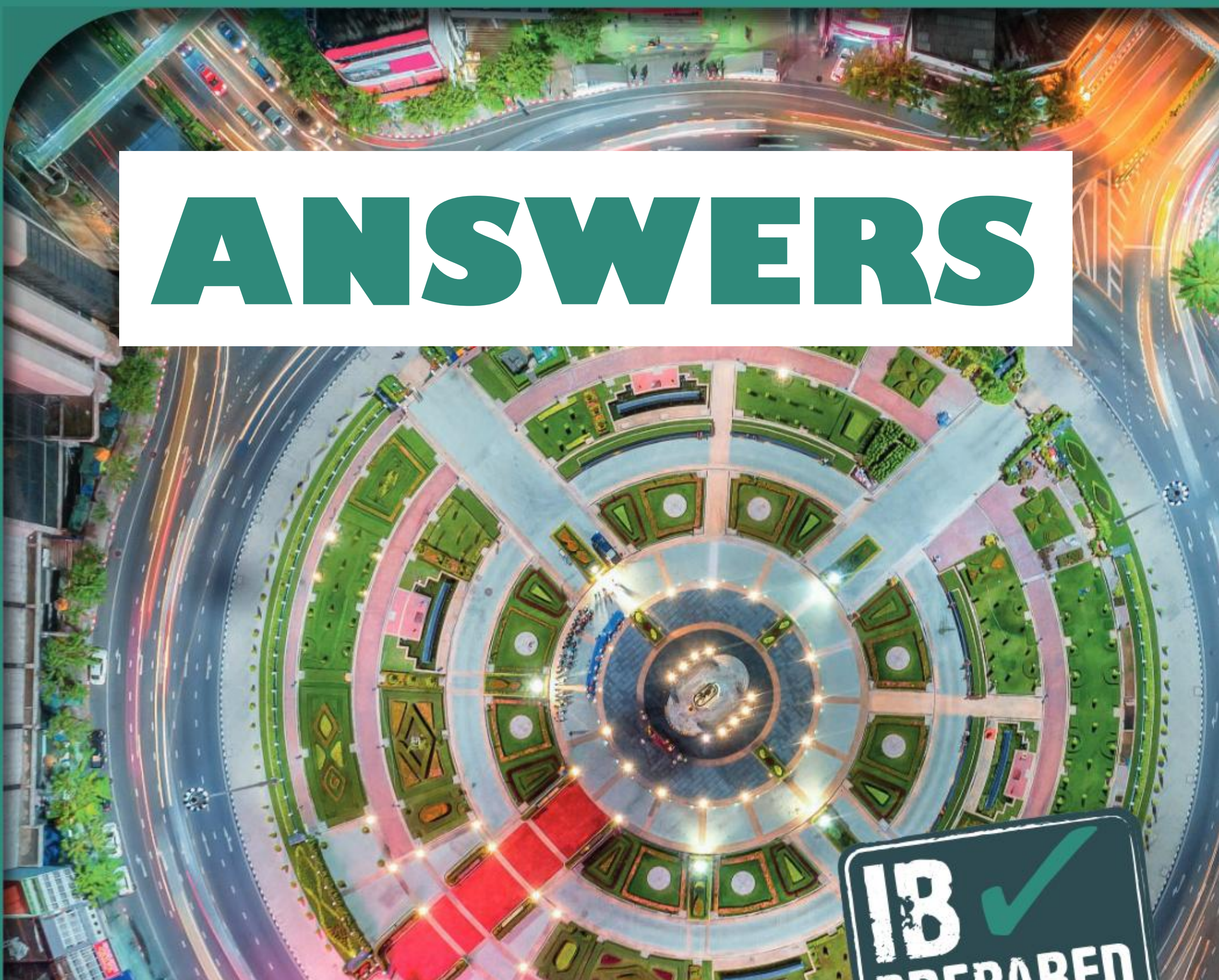
OXFORD IB PREPARED

MATHEMATICS: APPLICATIONS AND INTERPRETATION

ANSWERS



IB DIPLOMA PROGRAMME

Peter Gray
David Harris

OXFORD

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ANSWERS TO PRACTICE PROBLEMS

Here are the answers to the practice problems from IB Prepared Mathematics: Applications and Interpretations.

For direct access, click on the relevant paper.

Topic 4 Statistics and probability

SL Paper 1

SL Paper 2

HL Paper 1

HL Paper 2

HL Paper 3

Topic 5 Calculus

SL Paper 1

SL Paper 2

HL Paper 1

HL Paper 2

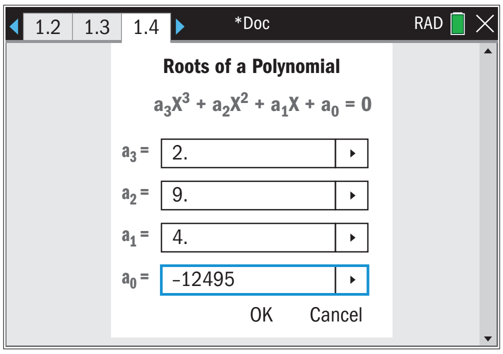
HL Paper 3

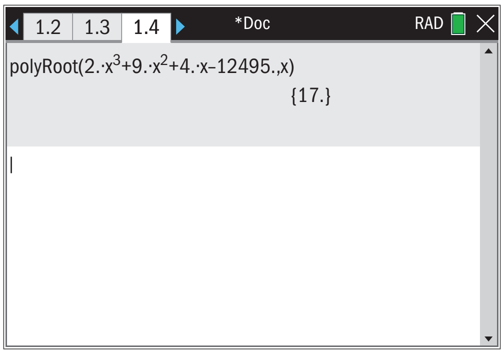
TOPIC 1 SL WORKED SOLUTIONS

Topic 1 SL Paper 1, Group 1

1. a. $V = n(n+4)(2n+1) = n\left(2n^2 + 9n + 4\right) = 2n^3 + 9n^2 + 4n$

b. $2n^3 + 9n^2 + 4n = 12\,495$ has solution $n = 17$ cm as shown below:





2. Let r_M be the measured radius of the hemisphere, and r be the true radius

a. $r_M = 6.3 \text{ cm} \Rightarrow 6.25 \text{ cm} \le r < 6.35 \text{ cm}$

Let V_M be the volume resulting from the measured radius, and V be the true radius.

b. $V_M = \dfrac{2}{3} \times \pi \times 6.3^3$ cm and

$\dfrac{2}{3} \times \pi \times 6.25^3 \text{cm} \le V < \dfrac{2}{3} \times \pi \times 6.35^3$ cm

Hence the maximum percentage error is $\left| \dfrac{\frac{2}{3} \times \pi \times 6.3^3 - \frac{2}{3} \times \pi \times 6.25^3}{\frac{2}{3} \times \pi \times 6.25^3} \right| \times 100\% = 2.42\%$

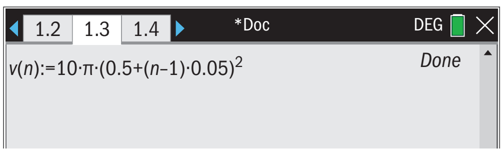
3. a. The fee in February 2026 will be $34 \times 1.02^2 = £35.37$ per month

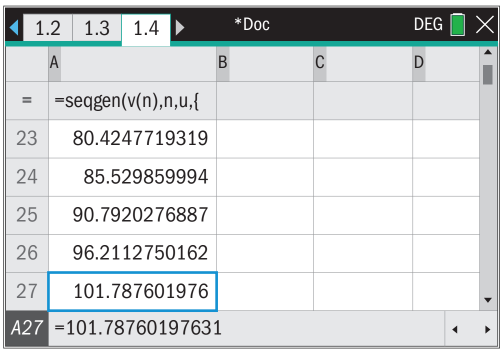
b. David will pay:

$34 \times 24 + 34 \times 1.02 \times 24 + 34 \times 1.02^2 \times 24 + 34 \times 1.02^3 \times 12 = £2930.26$

4. a. The columns have volumes in a sequence $V_n = 10\pi\left(0.5 + (n-1) \times 0.05\right)^2$

Using a spreadsheet, the first column to have a volume greater than 100 m^3 is column number 27.





b. Using sigma notation, the GDC can find the answer efficiently

$$\sum_{c=1}^{80} (v(c)) \qquad 18745.883364$$

The total volume of concrete needed is 18 746 m^3 to the nearest m^3.

5. a. $254 \times 65 \times 154 = 2\,542\,540 \text{ mm}^3 \approx 2\,500\,000 \text{ mm}^3$ (2 sf)

b. $2.5 \times 10^6 \text{ mm}^3$

c. The least the container holds is $253.5 \times 64.5 \times 153.5 = 2\,509\,840.125 \text{ mm}^3$

This is more than 2.5 litres since 2.5 litres = 2500 cm^3 = 2 500 000 mm^3 so the container can contain 2.5 litres of washing liquid.

6. a.

Finance Solver

N:	6
I(%):	4.3
PV:	-1600.0020732771
Pmt:	0.
FV:	2069.99
PpY:	1
CpY:	12
PmtAt:	END

Press ENTER to calculate
Number of Payments, N

He invests € 1600.00

b.

Finance Solver

N:	6.
I(%):	-12.460898314322
PV:	-2000.
Pmt:	0.
FV:	900.
PpY:	1
CpY:	1
PmtAt:	END

Finance Solver info stored into
tvm.n, tvm.i, tvm.pv, tvm.pmt, ...

The annual rate of depreciation is 12.5%

7. a. 252.65 million km = 252 650 000 km = 252 650 000 000 m = 2.5265×10^{11} m.

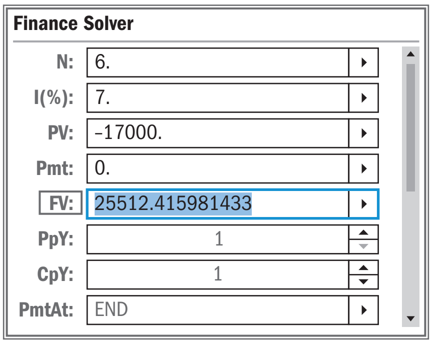
b. $60\,000 \text{ km h}^{-1} = \dfrac{60\,000 \times 1000}{60 \times 60} \text{ m s}^{-1} = 16\,666.67 \text{ m s}^{-1}$

In standard form, this is $1.67 \times 10^4 \text{ m s}^{-1}$

c. $\text{Time} = \dfrac{\text{distance}}{\text{speed}} = \dfrac{2.5265 \times 10^{11}}{1.666667 \times 10^4} = \dfrac{2.5265}{1.666667} \times 10^7 = 1.52 \times 10^7$ seconds.

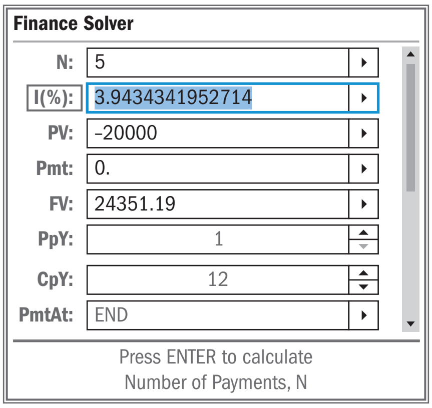
Topic 1 SL Paper 1, Group 2

8. a.



Pilar earns 25 512.42 – 17 000 = \$8512.42 USD interest.

b.



The interest rate of Ximena's account is 3.94%

9. a. $\log_{10}\left(\dfrac{52\,098}{0.001}\right) = 7.72$

b. $\log_{10}\left(\dfrac{x}{0.001}\right) = 8.9 \Rightarrow \dfrac{x}{0.001} = 10^{8.9} \Rightarrow x = 0.001 \times 10^{8.9} = 794\,328.234\,724\,28,$

which is 794 328 mm to the nearest mm.

10. a. $149\,600\,000 \times 1000 \times 100 = 14\,960\,000\,000\,000\,\text{cm}$

b. $14\,960\,000\,000\,000\,\text{cm} = 1.496 \times 10^{13}\ \text{cm}$

c. $2^x = 14\,960\,000\,000\,000 \Rightarrow x = \log_2 14\,960\,000\,000\,000 = 43.8$

Hence the value of p is 44.

11. a. This is an arithmetic sequence with first term $u_1 = 4$ and common difference $d = 0.5$.

Hence $u_{12} = 4 + 11 \times 0.5 = 9.5$ km in his 12th week of training.

b. $S_{20} = \sum_{r=1}^{20} (4 + (r-1) \times 0.5) = 175$ km

c. $u_n = 4 + (n-1) \times 0.5 \geq 42 \Rightarrow (n-1) \times 0.5 \geq 38 \Rightarrow n \geq 77$

Therefore at this rate Magzhan will first complete a full marathon during his 77th week of training.

12. **a.** $1624 = 1530 \times r^3 \Rightarrow r = \sqrt[3]{\dfrac{1624}{1530}} \approx 1.02$

The number of hires in 2025 is $1530 \times r^{10} = 1866$ hires.

b. $2000 = 1530 \times 1.02^{(n-1)} \Rightarrow n = 14.5$ so n must be 15 for the number of hires to exceed 2000.

Since $u_{15} = 1530 \times 1.02^{14}$, we require the 15th year in the sequence (where 2015 is the first year), which is 2029.

13. **a.** $17\,500 \times 0.95^7 = €12\,220.90$

b. $17\,500 \times 0.95^n = 0.4 \times 17\,500$ has solution $17.863\,752\,812\,425$ hence 18 complete years must pass.

Topic SL Paper 1, Group 3

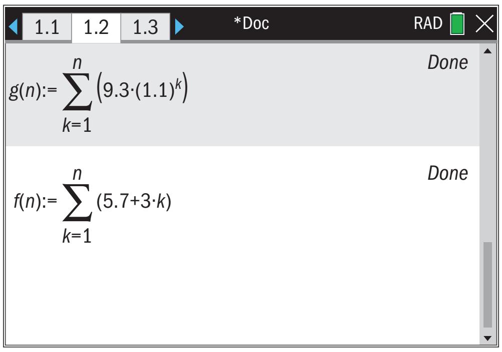
14. **a.** $g(n)$ represents an arithmetic series because $5.7 + 3k$ represents a series with a common difference.

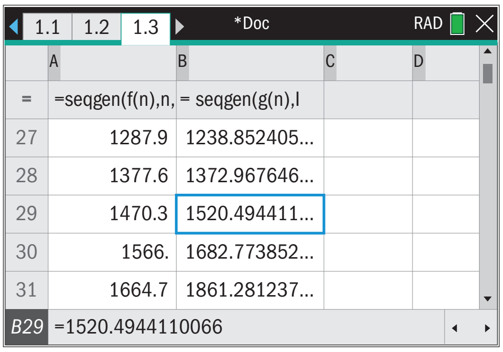
b. The first term is 8.7 and the common difference is 3.

c. $h(n)$ represents a geometric series because 9.3×1.1^k represents a series with a common ratio.

d. The first term is 10.23 and the common ratio is 1.1.

e.





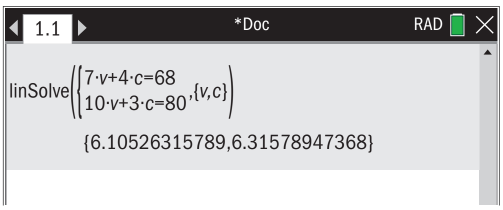
The solution $q = 29$ can be found from GDC as shown.

15. **a.** The context can be modelled with the system of equations

$7v + 4c = 68$

$10v + 3c = 80$

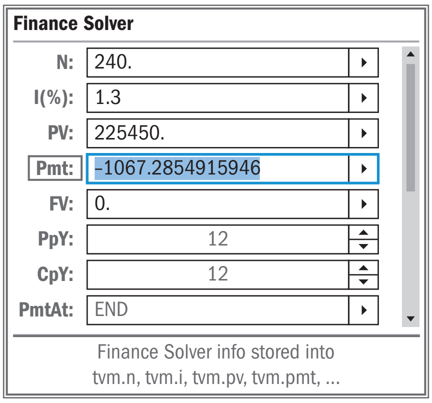
b. Use of GDC gives the solutions $v = 6.11$ and $c = 6.32$.

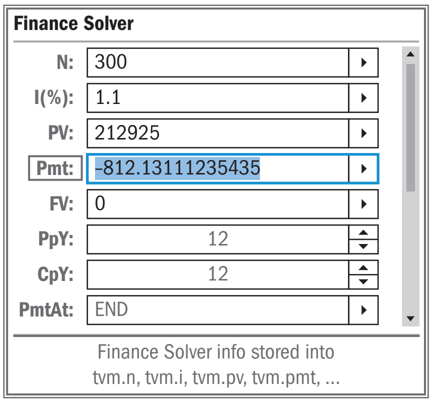


c. Hence $v = 6$ and $c = 6$ will not satisfy the requirements of the trip. $v = 7$ and $c = 6$ would mean 73 people and 88 cases can be transported, whereas $v = 6$ and $c = 7$ would mean 70 people and 81 cases can be transported.
This is the most economical solution to the problem.

16. a. Bank A requires a loan of $0.9 \times £250\,500 = £225\,450$ over 240 payments.

Bank B requires a loan of $0.85 \times £250\,500 = £212\,925$ over 300 payments.





The monthly payment for the Bank A deal is £1067.29 and for Bank B £812.13.

b. The total repayment for Bank A is £256 149.60 and for Bank B £243 639.00

c. For each bank, total expense = deposit + loan repayment
For Bank A, expense = 25 050 + 256 149.60 = 281 200.60
For Bank B, expense = 37 575 + 243 639 = 281 214.00

Advantages of choosing A: smaller deposit; the loan is paid off 5 years sooner than in B; total expense is slightly less.
Disadvantages of choosing A: The monthly payments are higher than in B.

17. a. $c = \sqrt{100 + 7 - 2(10)\sqrt{7}\,\cos(60^\circ)} = \sqrt{107 - 10\sqrt{7}}$

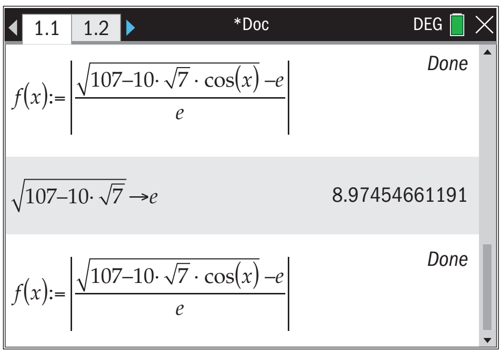
b. $\sqrt{7} \approx 2.65$ to three significant figures.

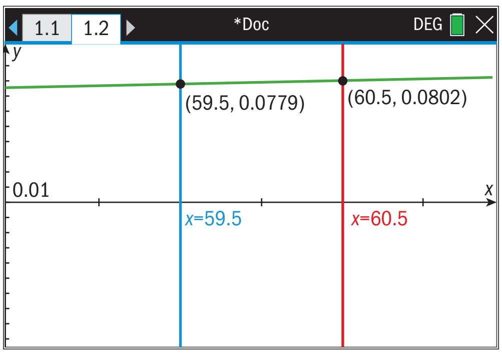
$$\left| \frac{\sqrt{100 + 2.65^2 - 2(10)(2.65)\cos(60^\circ)} - \sqrt{107 - 10\sqrt{7}}}{\sqrt{107 - 10\sqrt{7}}} \right| \times 100\% = 0.0124\%$$

c. $C = 60^\circ$ to the nearest degree so $59.5^\circ \le C_E < 60.5^\circ$

The percentage error on this domain is a function of C_E:

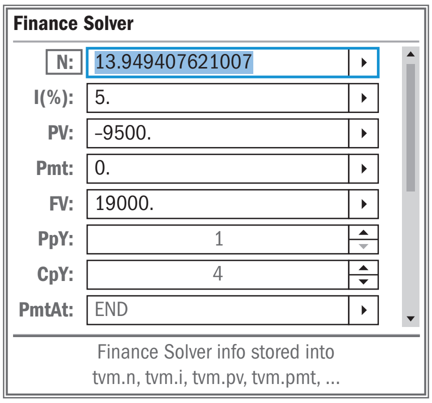
$$f(C_E) = \left| \frac{\sqrt{107 - 10\sqrt{7}\,\cos(C_E)} - \sqrt{107 - 10\sqrt{7}}}{\sqrt{107 - 10\sqrt{7}}} \right| \times 100\%$$





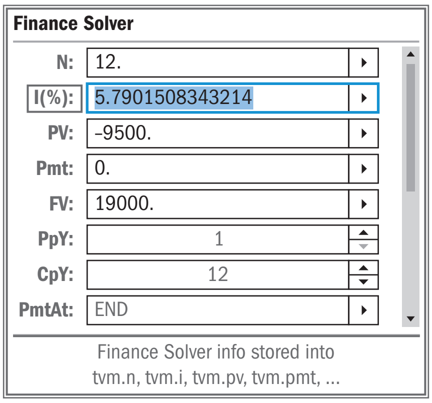
The maximum percentage error is therefore 0.0802 × 100% = 8.02%

18. a.



Chimdi must wait 14 years for her investment to double in value.

b.



Nicole's investment will double in value after 12 years with a nominal annual interest rate of 5.79%

Topic 1 SL Paper 2

1. a. $27 = y + 5x$.
$41 = y + 12x$

This is assuming that Denise increases the amount of subscriptions she sells each day by x.

b. $x = 2,\ y = 17$

c. If Denise sells 17 subscriptions on her first day and then increases the number of subscriptions she sells by just 1 each day for the 32 days, she will have sold $S_{32} = \frac{32}{2}(2\times 17 + (32-1)\times 1) = 16(34+31) = 1041$. This is the least she has sold, so she does sell more than 1000 subscriptions.

2. a. i. Option A: $21000(1.04) = €21\,840$

ii. Option B: $21000\left(1+\frac{3.8}{1200}\right)^{12} = €21\,812$

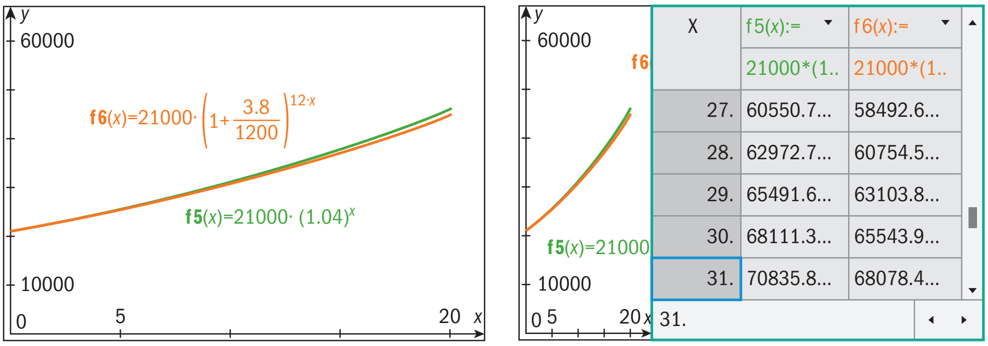
iii. Option C: $21000 + 1100 = €22\,100$

b. i. Option A: $V = 21000(1.04)^n$

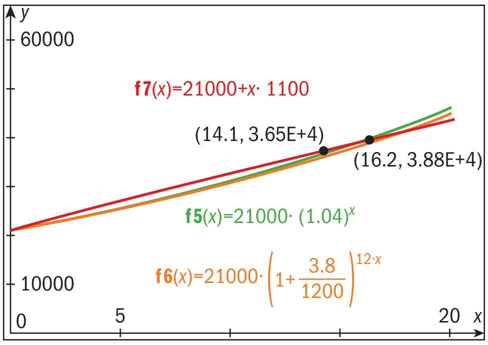
ii. Option B: $V = 21000\left(1+\frac{3.8}{1200}\right)^{12n}$

iii. Option C: $V = 21000 + 1100n$

c. Using the GDC to investigate, it is found that the value of Option A is always more than the value of Option B. So Beth's claim is false since each Option starts with €21 000.



d.



The GDC shows that after 17 years, Alex's investment would be worth more in both Option A and B than in C.

e. After 20 years in Option A, Alex's investment is worth

$$21000(1.04)^{20} = €46\,014$$

After 20 years in Option C, Alex's investment is worth

$$21\,000 + (20)(1100) = €43\,000$$

The difference in the interest earned is therefore €3014.

TOPIC 1 HL WORKED SOLUTIONS

Topic 1 HL Paper 1, Group 1

1. a. $\begin{pmatrix} 2 & 4 \\ 1.5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} 2-\lambda & 4 \\ 1.5 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left|\begin{pmatrix} 2-\lambda & 4 \\ 1.5 & 1-\lambda \end{pmatrix}\right| = 0$

$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$ which when solved gives the eigenvalues 4 and –1.

If $\lambda = 4$ then $\begin{pmatrix} -2 & 4 \\ 1.5 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -2x+4y=0 \\ 1.5x-3y=0 \end{matrix} \Rightarrow x = 2y$

If $\lambda = -1$ then $\begin{pmatrix} 3 & 4 \\ 1.5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 3x+4y=0 \\ 1.5x+2y=0 \end{matrix} \Rightarrow -3x = 4y$

with associated eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ respectively.

b. $\mathbf{C} = \begin{pmatrix} 2 & 4 \\ 1.5 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}^{-1}$

c. $\mathbf{C}^n = \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 0.3 & 0.4 \\ -0.1 & 0.2 \end{pmatrix}$

2. a. $R = \log_{10}\left(\frac{I}{I_0}\right) \Rightarrow \frac{I}{I_0} = 10^R \Rightarrow I = I_0 10^R.$

Hence the ratio of the intensity of the San Francisco earthquake to that of the Dashur earthquake is $\frac{I_0 10^{8.25}}{I_0 10^{5.9}} = 10^{2.35} \approx 224.$

Hence the San Francisco earthquake was 224 times more intense than that of Dashur.

b. Let J represent the measure of the Japanese earthquake on the Richter scale and $J = \log_{10}\left(\frac{I}{I_0}\right) \Rightarrow I = I_0 10^J = 7.08(I_0 10^{8.25}) = 10^{\log_{10} 7.08} I_0 10^{8.25}.$

$I_0 10^J = 10^{\log_{10} 7.08} I_0 10^{8.25} \Rightarrow 10^J = 10^{8.25 + \log_{10} 7.08}$

Therefore $J = 8.25 + \log_{10} 7.08 \approx 9.10.$

3. a. 0.151 515 151 515 =

0.15 + 0.0015 + 0.000 015 + 0.000 000 15 + 0.000 000 000 015 + …

which is a geometric series first term 0.15, common ratio 0.01 and 6 terms.

Hence

$$\sum_{r=1}^{6} 0.15 \times (0.01)^{n-1} = \sum_{r=1}^{6} 0.15 \times \left(10^{-2}\right)^{n-1} = \sum_{r=1}^{6} 0.15 \times 10^{2-2n}$$

b. $0.\overline{51} = 0.515151515151 = 0.51 + 0.0051 + 0.000051 + \ldots$

$= 0.51 + 0.51 \times 0.001 + 0.51 + 0.00001 + \ldots$

$0.\overline{51} = \sum_{r=1}^{\infty} 0.51 \times (0.01)^{r-1}$

c. $0.\overline{51} = \frac{0.51}{1-0.01} = \frac{51}{99} = \frac{17}{33}$

4. $\frac{z}{z+2\mathrm{i}} = 4-7\mathrm{i} \Rightarrow z = (z+2\mathrm{i})(4-7\mathrm{i}) \Rightarrow z = 4z - 7\mathrm{i}z + 8\mathrm{i} + 14$

$\Rightarrow z(-3+7\mathrm{i}) = (14+8\mathrm{i})$

Hence $z = \frac{14 + 8\mathrm{i}}{-3 + 7\mathrm{i}} = \frac{7}{29} - \frac{61}{29}\mathrm{i}$

5. a. The roots of $3x^2 + x - 7 = 0$ are $x_1, x_2 = \frac{-1 \pm \sqrt{1^2 - 4(3)(-7)}}{2(3)} = \frac{-1 \pm \sqrt{85}}{6}$,

so that $x_1, x_2 = \frac{-1+\sqrt{85}}{6}, \frac{-1-\sqrt{85}}{6}$

b. The roots of $3x^2 + x + 7 = 0$ are $z_1, z_2 = \frac{-1 \pm \sqrt{1^2 - 4(3)(7)}}{2(3)} = \frac{-1 \pm \sqrt{-83}}{6}$, so that

$z_1, z_2 = \frac{-1+\sqrt{83}\mathrm{i}}{6}, \frac{-1-\sqrt{83}\mathrm{i}}{6}$

c. $\frac{-1+\sqrt{85}}{6} + \frac{-1-\sqrt{85}}{6} = \frac{-2}{6} = \frac{-1+\sqrt{83}\mathrm{i}}{6} + \frac{-1-\sqrt{83}\mathrm{i}}{6}.$

For both quadratic equations $3x^2 + x - 7 = 0$ and $3x^2 + x + 7 = 0$, it follows that $\frac{-b}{a} = \frac{-1}{3}$ so the conjecture is true.

6. a. $\sqrt[3]{9x^{14} \times 375x^{10} \times (2x^2)^3} = \sqrt[3]{9x^{14} \times 375x^{10} \times 8x^6} = \sqrt[3]{x^{30} \times 3^3 \times 2^3 \times 5^3} = 30x^{10}$

b. $\left(\frac{729x^{12}}{64y^3}\right)^{\frac{1}{3}} = \frac{729^{\frac{1}{3}}(x^{12})^{\frac{1}{3}}}{(64y^3)^{\frac{1}{3}}} = \frac{9x^4}{4y}$

c. $\frac{(5y^{-1}x^2)^3}{(3y^2x^{-4})^5} = \frac{125y^{-3}x^6}{243y^{10}x^{-20}} = \frac{125}{243}y^{-13}x^{26}$

Topic 1 HL Paper 1, Group 2

7. a. $\omega_1\omega_2 = 6\mathrm{e}^{\frac{\pi}{12}\mathrm{i}}$

b. $\left(\frac{\omega_1}{\omega_2}\right)^4 = \frac{16\operatorname{cis}(\pi)}{81\operatorname{cis}\left(\frac{-2\pi}{3}\right)} = \frac{16}{81}\mathrm{e}^{\frac{5\pi}{3}\mathrm{i}}$

c. $(\omega_1^*)^3 = \left(2\operatorname{cis}\left(\frac{-\pi}{4}\right)\right)^3 = 8\mathrm{e}^{-\frac{3\pi}{4}\mathrm{i}}$

8. $V = 7\sin(20t+5) + 3\sin(20t+13)$

$$= \mathrm{Im}\left(7e^{(20t+5)i}\right) + \mathrm{Im}\left(3e^{(20t+13)i}\right)$$

$$= \mathrm{Im}\left(7e^{(20t+5)i} + 3e^{(20t+13)i}\right)$$

$$= \mathrm{Im}\left(e^{20ti}\left(7e^{5i} + 3e^{13i}\right)\right)$$

$$= \mathrm{Im}\left(e^{20ti}\left(7.20e^{-0.858i}\right)\right)$$

$$= \mathrm{Im}\left(7.20e^{(20t-0.858)i}\right)$$

$$= 7.20\sin(20t - 0.858)$$

Hence the maximum value of V is 7.20 and the phase shift is –0.858.

9. a. $\mathbf{XA} + \mathbf{XB} = \mathbf{C} - 3\mathbf{D}$

$$\Rightarrow \mathbf{X}(\mathbf{A}+\mathbf{B}) = \mathbf{C} - 3\mathbf{D}$$

$$\Rightarrow \mathbf{X} = (\mathbf{C} - 3\mathbf{D})(\mathbf{A}+\mathbf{B})^{-1}$$

$$= \begin{pmatrix} -12 & \pi \\ 6 & -3\sqrt{3} \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -12 & \pi \\ 6 & -3\sqrt{3} \end{pmatrix} \times \frac{-1}{13}\begin{pmatrix} 1 & -3 \\ -5 & 2 \end{pmatrix}$$

$$= \frac{-1}{13}\begin{pmatrix} -12-5\pi & 36+2\pi \\ 6+15\sqrt{3} & -18-6\sqrt{3} \end{pmatrix}$$

b. $\mathbf{BXA} = \mathbf{D} \Rightarrow \mathbf{BX} = \mathbf{DA}^{-1} \Rightarrow \mathbf{X} = \mathbf{B}^{-1}\mathbf{DA}^{-1} = \begin{pmatrix} 0 & 5 \\ 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 4 & 0 \\ -3 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 3 & 0 \end{pmatrix}^{-1}$

$$= \frac{1}{-10}\begin{pmatrix} 1 & -5 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ -3 & \sqrt{3} \end{pmatrix}\frac{1}{6}\begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix} = \frac{1}{-60}\begin{pmatrix} 19 & -5\sqrt{3} \\ -8 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix}$$

$$= \frac{1}{-60}\begin{pmatrix} 15\sqrt{3} & 38-10\sqrt{3} \\ 0 & -16 \end{pmatrix}$$

10. a. $-1 < 1.5\left(2^x\right) < 1 \Rightarrow \frac{-2}{3} < 2^x < \frac{2}{3}$. But the range of the exponential function is only positive numbers, so $0 < 2^x < \frac{2}{3}$, therefore $x < \log_2 \frac{2}{3}$.

b. $S_\infty = \dfrac{13}{1-1.5\left(2^x\right)} = 16 \Rightarrow 13 = 16 - 24(2^x) \Rightarrow 2^x = \dfrac{1}{8} \Rightarrow x = -3.$

11. a. The common difference d is $\log_{10}(a+9) - \log_{10} a = \log_{10}(a+20) - \log_{10}(a+9)$

Hence $\log_{10}\left(\dfrac{a+9}{a}\right) = \log_{10}\left(\dfrac{a+20}{a+9}\right)$, therefore $\dfrac{a+9}{a} = \dfrac{a+20}{a+9}$ so that

$$(a+9)^2 = a(a+20)$$

$$a^2 + 18a + 81 = a^2 + 20a \Rightarrow 2a = 81 \Rightarrow a = 40.5$$

b. $u_1 = \log_{10}(40.5),\ d = \log_{10}\left(\frac{49.5}{40.5}\right)$

$$S_{20} = 10\left(2\log_{10}(40.5) + 19(\log_{10}\left(\frac{49.5}{40.5}\right)\right) \approx 48.7$$

12. a. $\omega = \text{cis}\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) + \text{i}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}.$

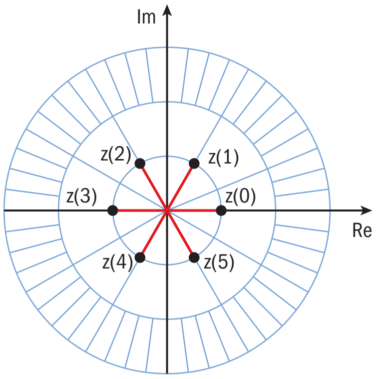
b. $\omega^2 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}\right)^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}$ in cartesian form

$\omega^2 = \text{cis}\left(\frac{2\pi}{3}\right)$ in modulus-argument form.

Since $\text{cis}\left(\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) + \text{i}\sin\left(\frac{2\pi}{3}\right)$, $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ and $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$ by equating real and imaginary parts.

c. $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5\} = \left\{1, \text{cis}\left(\frac{\pi}{3}\right), \text{cis}\left(\frac{2\pi}{3}\right), \text{cis}(\pi), \text{cis}\left(\frac{4\pi}{3}\right), \text{cis}\left(\frac{5\pi}{3}\right)\right\}$

d.



Applying symmetry in the Argand diagram, the vectors representing $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5\}$ have resultant 0 when added, since

$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5$

$= 1 + \frac{1}{2} + \frac{\sqrt{3}}{2}\text{i} + \frac{-1}{2} + \frac{\sqrt{3}}{2}\text{i} + (-1)$

$+ \frac{-1}{2} + \frac{-\sqrt{3}}{2}\text{i} + \frac{1}{2} + \frac{-\sqrt{3}}{2}\text{i} = 0.$

Topic 1 HL Paper 1, Group 3

13. a. $\log_{10}(ax+b) = 2 + 2\log_{10}(ax-b)$

$\Rightarrow \log_{10}(ax+b) = \log_{10}100 + \log_{10}(ax-b)^2$

$\Rightarrow \log_{10}(ax+b) = \log_{10}100(ax-b)^2 \Rightarrow 100(ax-b)^2 = (ax+b)$

Hence $100a^2x^2 - 200axb + 100b^2 - ax - b = 0$ so

$p = 100a^2$, $q = -200a$ and $r = 100b^2 - b$

b. $\ln(2-r) = 3 + 2(2)^3 = 19 \Rightarrow r = 2 - \text{e}^{19}$. $\text{e}^s = 3 + 2(1)^3 = 5 \Rightarrow s = \ln(5)$

c. $4^{2x+1} = 3^{1-x} \Rightarrow (2x+1)\ln(4) = (1-x)\ln(3) \Rightarrow 2\ln(4)x + \ln(4) = \ln(3) - \ln(3)x$

$$\Rightarrow x(2\ln(4) + \ln(3)) = \ln(3) - \ln(4) \Rightarrow x(\ln(16) + \ln(3)) = \ln\left(\frac{3}{4}\right) \Rightarrow x = \frac{\ln\left(\frac{3}{4}\right)}{\ln(48)}$$

14. a. $zw = \left(2 + 2\sqrt{3}\text{i}\right)\left(\sqrt{2} - \sqrt{2}\text{i}\right) = \left(2\sqrt{2} + 2\sqrt{6}\right) + \text{i}\left(2\sqrt{6} - 2\sqrt{2}\right)$

$$\frac{z}{w} = \frac{\left(2 + 2\sqrt{3}\text{i}\right)}{\left(\sqrt{2} - \sqrt{2}\text{i}\right)} = \frac{\left(2 + 2\sqrt{3}\text{i}\right)\left(\sqrt{2} + \sqrt{2}\text{i}\right)}{\left(\sqrt{2} - \sqrt{2}\text{i}\right)\left(\sqrt{2} + \sqrt{2}\text{i}\right)} = \frac{\left(2\sqrt{2} - 2\sqrt{6}\right) + \text{i}\left(2\sqrt{6} + 2\sqrt{2}\right)}{4} = \frac{\left(\sqrt{2} - \sqrt{6}\right)}{2} + \frac{\text{i}\left(\sqrt{6} + \sqrt{2}\right)}{2}$$

b. $|z| = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = 4$ $\quad$ $|w| = \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} = 2$

$\arg(z) = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$ $\quad$ $\arg(w) = \arctan(-1) = \frac{-\pi}{4}$

Hence $z = 4e^{\frac{\pi}{3}i}$ and $w = 2e^{\frac{-\pi}{4}i} \Rightarrow zw = 8e^{\frac{\pi}{12}i}$ and $\frac{z}{w} = 2e^{\frac{7\pi}{12}i}$

c. $\left(2\sqrt{2} + 2\sqrt{6}\right) + i\left(2\sqrt{6} - 2\sqrt{2}\right) = 8e^{\frac{\pi}{12}i} = 8\left(\cos\left(\frac{\pi}{12}\right) + i\left(\sin\left(\frac{\pi}{12}\right)\right)\right)$ hence by comparing real and imaginary parts, $\cos\left(\frac{\pi}{12}\right) = \frac{\left(2\sqrt{2} + 2\sqrt{6}\right)}{8} = \frac{\left(\sqrt{2} + \sqrt{6}\right)}{4}$.

15. a.

$$\mathbf{RP} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{pmatrix} 29 & 35 \\ 19 & 21 \\ 8 & 12 \end{pmatrix} \end{array}$$

RP gives the number of points that the teams would score under each proposed points system

b. $\mathbf{XS} = \mathbf{E}$ has solution $\mathbf{X} = \mathbf{ES}^{-1}$. Hence $\mathbf{X} = \begin{pmatrix} 12000 & 15600 \\ 10000 & 20000 \\ 8000 & 14000 \end{pmatrix} \times \begin{pmatrix} 70 & 65 \\ 40 & 100 \end{pmatrix}^{-1}$.

Using a GDC, this gives the solution $\mathbf{X} = \begin{pmatrix} 131 & 71.0 \\ 45.5 & 170 \\ 54.5 & 105 \end{pmatrix}$ where each element is correct to three significant figures.

This matrix represents the number of points that each time would have to score in order to gain the amount of sponsorship money desired by each team.

16. $1 + 2\log_{10} y = \log_{10} 10x \Rightarrow \log_{10} 10 + \log_{10} y^2 = \log_{10} 10x$ hence $10y^2 = 10x$

$1 + \log_{10} x = \log_{10}(7 - 20y) \Rightarrow \log_{10} 10 + \log_{10} x = \log_{10}(7 - 20y)$ hence $10x \Rightarrow 7 - 20y$

So $7 - 20y = 10y^2 \Rightarrow 10y^2 + 20y - 7 = 0$ so $(3y - 1)(y + 7) = 0$.

This equation has solutions $y = -2.30$ and $y = 0.304$. However, $y = -2.30$ does not fit the original system so the only solution to the system is $y = 0.304$, $x = 0.0923$.

17. $\ln\left(\frac{y^2 z}{x}\right) = 3 \Rightarrow -\ln(x) + 2\ln(y) + \ln(z) = 3$

$\ln\left(x^2\right) + \ln\left(y^3\right) = 10 \Rightarrow 2\ln(x) + 3\ln(y) = 10$

$\ln\left(\sqrt{xyz}\right) = -1 \Rightarrow 0.5\ln(x) + 0.5\ln(y) + 0.5\ln(z) = -1$

Using a GDC to solve the system of linear equations:

$\ln(x) = -0.625$, $\ln(y) = 3.75$, $\ln(z) = -5.125$

Hence $x = e^{-0.625}$, $y = e^{3.75}$ and $z = e^{-5.125}$.

18. a. The completed matrix **P** is

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

b. Hence $\mathbf{P}^2$ is

$$\begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 3 & 3 & 2 \\ 1 & 3 & 6 & 1 \\ 3 & 2 & 1 & 5 \end{pmatrix}$$

c. There are 3 two-stage journeys from C to B:

$C \to D \to B$ via two different routes from C to D, and $C \to A \to B$.

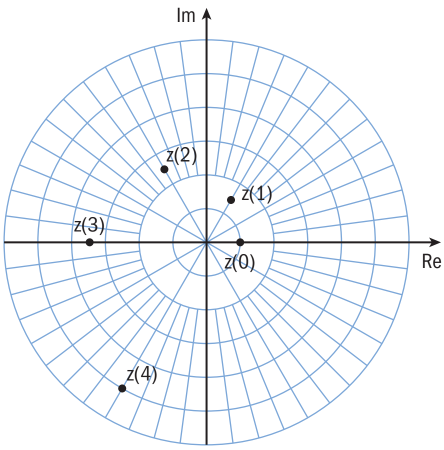
This is represented in the element in Row 3, column 2 of $\mathbf{P}^2$.

d. The elements of $\mathbf{P}^2$ give the number of two-stage journeys between each of the four towns.

Topic 1 HL Paper 2

1. a. $z(0)=1,\ z(1)=1.5e^{i\frac{\pi}{3}},\ z(2)=2.25e^{i\frac{2\pi}{3}},\ z(3)=-3.375,\ z(4)=5.0625e^{i\frac{4\pi}{3}}$

b.



c. In sector D, the argument of a complex number is between 0 and $-\frac{\pi}{2}$.

The first complex number in the sequence to have modulus greater than 50 and argument between 0 and $-\frac{\pi}{2}$ is $z(11)=86.5e^{-1.05i}$.

Hence, after 11 hours the UAV is first more than 50 km from its initial position **and** is in sector D.

d. $z(13)=1.5^{13}e^{\frac{13\pi}{3}i}=1.5^{13}e^{\frac{\pi}{3}i}=1.5^{13}\left(\cos\left(\frac{\pi}{3}\right)+i\left(\sin\left(\frac{\pi}{3}\right)\right)\right)=1.5^{13}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$z(0)=1+0i$. Hence the UAV travels on the vector $\begin{pmatrix} 1-\frac{1.5^{13}}{2} \\ -\frac{1.5^{13}\sqrt{3}}{2} \end{pmatrix}$ in order to return to its starting position.

e. At $t = 6$ hours, the arguments of $z(6)$ and $a(6)$ are 2π and 4π respectively. The modulus of $z(6)$ is 11.3906... km = 11 391 m to the nearest metre, and the modulus of $a(6)$ is 11.8539... km = 11 854 m to the nearest metre.

Although the two UAVs have the same argument at $t = 6$ hours, they are 463 metres apart at this time, and this does not qualify as a "near miss"

2. a.

$$\mathbf{E} = \begin{array}{l} \\ \text{IT} \\ \text{sales} \\ \text{office} \end{array} \begin{array}{c} \begin{array}{cc} \text{Cai} & \text{Tls} \end{array} \\ \begin{pmatrix} 52 & 34 \\ 15 & 9 \\ 19 & 12 \end{pmatrix} \end{array}$$

b. i. $\mathbf{S}$ would represent the salaries of IT, sales and office employees respectively.

ii. The dimensions of $\mathbf{S}$ are 1×3.

c. $x = 42\,900$

d. x represents the total salaries paid to the Cairo *BetaGamma* employees in one week.

e. $52 \times (66\,800 + 42\,900) = \text{USD\$}5\,704\,400$

f. $1.01^2 \times \begin{pmatrix} 900 & 700 & 500 \end{pmatrix} \times \begin{pmatrix} 50 & 32 \\ 13 & 7 \\ 17 & 10 \end{pmatrix} = \begin{pmatrix} 63858.26 & 39477.87 \end{pmatrix}$

The total paid to all *BetaGamma* employees in one year is

$52 \times 103\,336.13 = \text{USD\$}5\,373\,478.76$

Topic 1 HL Paper 3

1. a. i. $\frac{7}{6}$

ii. 0.5

iii. $\frac{1}{3}$

b. 64 pups, 10 young and 37 adults

c. Let the initial population vector be $\boldsymbol{q} = \begin{pmatrix} 20 \\ 15 \\ 40 \end{pmatrix}$

After one year the population is $\mathbf{L}\boldsymbol{q}$

After two years it is $\mathbf{L}(\ \mathbf{L}\boldsymbol{q}\ ) = \mathbf{L}^2\boldsymbol{q}$

After three years is $\mathbf{L}(\ \mathbf{L}^2\ \boldsymbol{q}\ ) = \mathbf{L}^3\ \boldsymbol{q}$

d. i. $\begin{pmatrix} 0 & \frac{7}{6} & \frac{7}{6} \\ 0.5 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}^{20} \begin{pmatrix} 20 \\ 15 \\ 40 \end{pmatrix} = \begin{pmatrix} 971 \\ 416 \\ 555 \end{pmatrix}$

$T_{20} = 1942$

ii. $\begin{pmatrix} 4537 \\ 1944 \\ 2593 \end{pmatrix}$

$T_{30} = 9074$

e. i. $T_{30} = T_{20} r^{10}$

$r^{10} = \dfrac{9074}{1942}$

$= 4.67$ (3 sf)

$r = 4.67^{\frac{1}{10}} = 1.167$

ii. $5000 = 9074 \times 1.17^{n-30}$

41 years

f. $q_{30} = \dfrac{1}{9074}\begin{pmatrix} 4537 \\ 1944 \\ 2593 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.2142\ldots \\ 0.2857\ldots \end{pmatrix}$

g. i. $\begin{pmatrix} 0 & \frac{7}{6} & \frac{7}{6} \\ 0.5 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 0.5 \\ 0.2142\ldots \\ 0.2857\ldots \end{pmatrix} = \lambda\begin{pmatrix} 0.5 \\ 0.2142\ldots \\ 0.2857\ldots \end{pmatrix}$

Form any equation for example

$0.5 \times 0.5 = \lambda \times 0.2142\ldots$

$\lambda = 1.166\ldots$

ii. This is equal to the common ratio for the increase in the population size.

TOPIC 2 SL WORKED SOLUTIONS

Topic 2 SL Paper 1, Group 1

1. **a.** $M = (2,3)$

b. The gradient of CM is $\dfrac{8-3}{0-2} = -2.5$

c. $y - 3 = -2.5(x-2) \Rightarrow y = -2.5x + 8$

This can be written as $5x + 2y - 16 = 0$

2. **a.** $I \sim \dfrac{1}{d^2}$ so $I = \dfrac{k}{d^2}$.

Hence $97 = \dfrac{k}{32^2}$. $k = 97 \times 32^2 = 99\,328$.

The equation relating I and d is therefore $I = \dfrac{99\,328}{d^2}$.

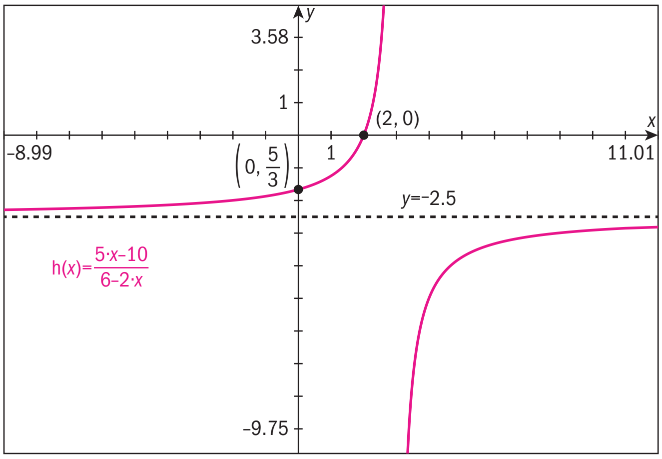
b. $I = \dfrac{99\,328}{6.7^2} = 2210$ cd

3. **a.** The x-coordinate of the x-intercept is found by solving $h(x) = 0$. Hence $5x - 10 = 0$ which gives $x = 2$. Hence the coordinates of the x-intercept are $(2, 0)$. The y-coordinate of the y-intercept is found by finding $h(0)$. Since $h(0) = \dfrac{-10}{6} = -\dfrac{5}{3}$, the coordinates of the y intercept are $\left(0, -\dfrac{5}{3}\right)$.

b. Using technology to consider values of h for large positive and negative values of x, the limiting value is -2.5 so the equation of the horizontal asymptote is $y = -2.5$

c. The largest possible domain is not the set of real numbers because $x = 3$ has no image because the denominator of the function would be equal to zero.

d. Your sketch should show the following features, equations and shape:



4. **a.** The gradient of L_1 is $\dfrac{1}{5}$

b. $x = 25$

c. The gradient of L_2 is -5

$y - 4 = -5(x - 0) \Rightarrow y = -5x + 4$.

5. Design A has slope $\frac{-5}{56} \approx 0.0892$. Design B has slope $\frac{4}{50} = 0.08$. Hence both designs fit the safety regulations

6. a. $4.53 = 231a + b$

   b. $4.71 = 301a + b$

   c. Using the GDC to solve the system:

```
linSolve({4.53=231·a+b, 4.71=301·a+b},{a,b})
                         {0.002571428571,3.936}
276·0.002571428571+3.936
                                  4.6457142856
```

   The length of the girder at 276 °C is 4.65 m

Topic 2 SL Paper 1, Group 2

7. a. $99328 = R \times 2^5 \Rightarrow R = 3104$

   b. R is the number of people who view the post the moment it was made.

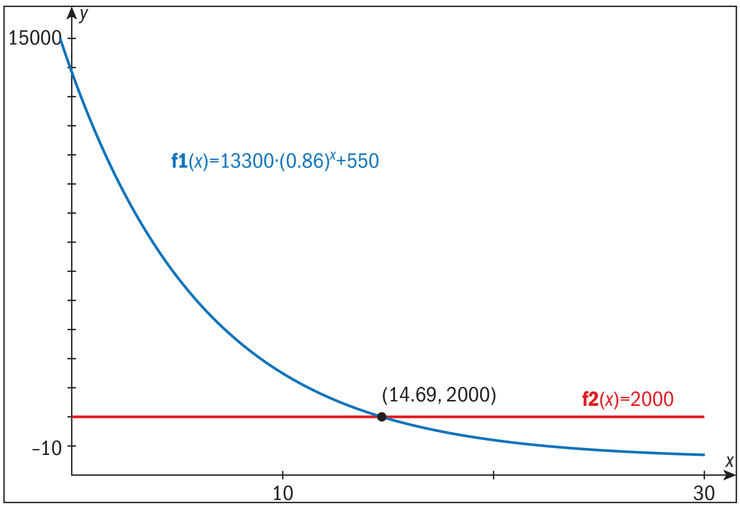
   c. $1500R = 4656000$

   $M(10) = 3104 \times 2^{10} = 3178496 < 4656000$, so the number of people viewing the post will not exceed $1500R$ in the domain given.

   d. If the domain was all real numbers this would not be appropriate since the range would include infinitely many numbers that are larger than the population of the world.

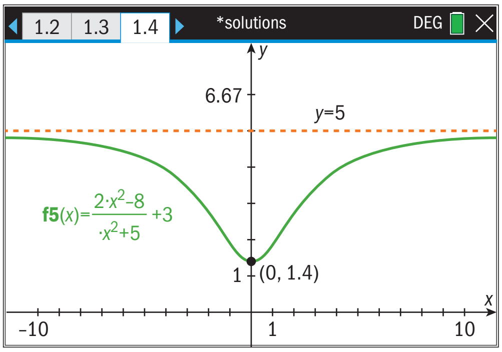
8. a. The value of David's car is €5930 after 6 years.

   b. After 15 years, David will be able to exchange his car for the discount.



   c. €550

9. a.



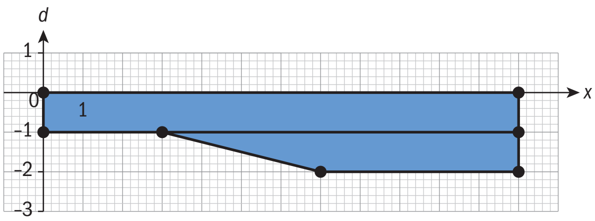
b. i. $P(3) = 3700$ to the nearest hundred.

ii. The population of the island will approach 5000 people in the long term but never exceed this value.

10. a.

x	1	3	5.5	12
Depth	-1	-1	-1.625	-2

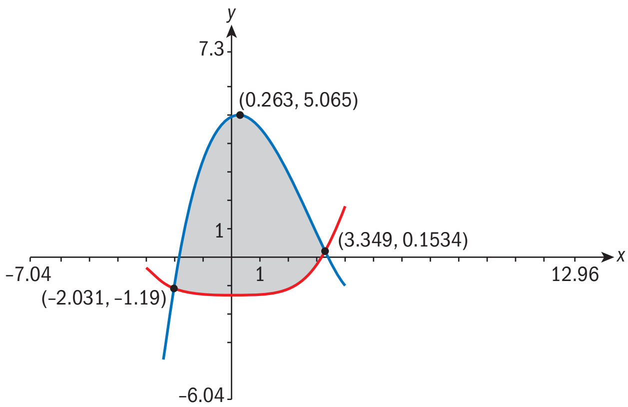
b.



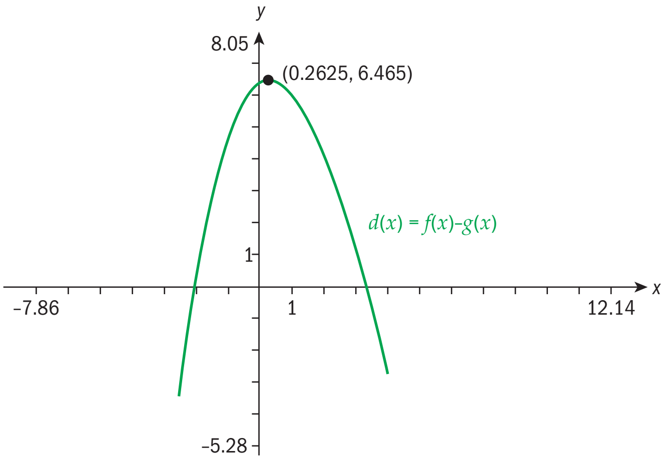
The swimming pool diagram can be separated into a rectangle with area 12 m^2 and a trapezium with area $\frac{1}{2}(1)(5+9)=7\text{ m}^2$. Hence the volume is 57 m^3.

11. a. To find the range of f, find the maximum value with technology and also find $f(-2.5)$. Hence the range is $-4.453125 \le y \le 5.065$

b. The greatest possible difference between the x-coordinates of any two points in the pool is 3.349 + 2.031 = 5.38 m



c.



The graph shows that the greatest possible difference between the y-coordinates of any two points in the pool is 6.465 m.

Topic 2 SL Paper 1, Group 3

12. a. $V \sim x^3$ so $V = kx^3$

$7.36 = k(1.5^3)$ hence $k \approx 2.18$. Hence $V = 2.18x^3$

b. $V = 2.18x^3 = 2.18 \times 5^3 = 272.5\ \text{m}^3$

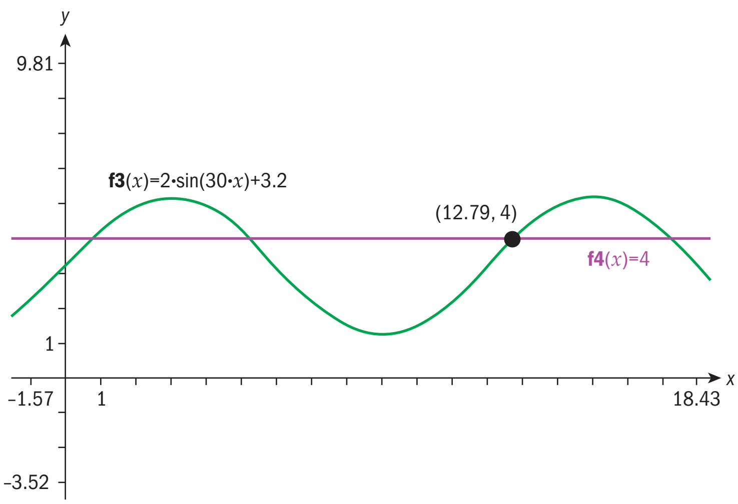
c. $0.45 \le V < 0.55 \Rightarrow 0.45 \le 2.18x^3 < 0.55$

Hence $\sqrt[3]{\dfrac{0.45}{2.18}} \le x < \sqrt[3]{\dfrac{0.55}{2.18}}$, giving $0.591 \le x < 0.632$

13. a. $p = \dfrac{5.2 - 1.2}{2} = 2,\ r = \dfrac{5.2 + 1.2}{2} = 3.2,\ s = \dfrac{360}{12} = 30$

b. r is the average depth of the water in a 12-hour period.

c.



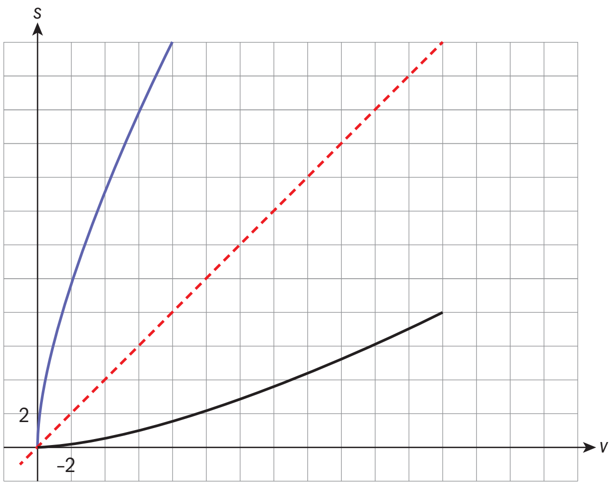
George can sail at 12:48 since $0.79 \times 60 = 47.4$

14. a. $S(0) = 0$

b. $S(8) = 6\left(\sqrt[3]{8}\right)^2 = 24$. This means that a cube with volume 8 cm^3 has surface area 24 cm^2.

c. $p = 0$ and $q = 24$

d.



e. $S^{-1}(6)=1$ means that a cube with surface area 6 cm^2 has volume 1 cm^3.

15. **a.** $\left.\begin{aligned} p+q+r&=143\\ 49p+7q+r&=107\\ 81p+9q+r&=63 \end{aligned}\right\}$ hence $p=-2,\ q=10$ and $r=135$

b. The height of the tower is 135 m.

c. The football is at its maximum height at $\left(\frac{-10}{2\times(-2)},\ H\left(\frac{-10}{2\times(-2)}\right)\right)=(2.5,\ 147.5)$.

d. The football reaches the ground at the point (11.1, 0).

16. **a.** $c=25$

b. $a(-5)^2+4(-5)+25=0$, hence $a=\frac{-1}{5}$ so $h(x)=\frac{-1}{5}x^2+4x+25$

c. The height of the roof is $h\left(\frac{-4}{2\times\frac{-1}{5}}\right)=45$ m.

d. A rectangular area of 1400 m^3 with length 32 m has height 43.75 m hence the roof would not fit in such a shape.

17. **a.** The midpoint of HF is $M=(9, 11)$. The gradient of HF is $\frac{-18}{14}=\frac{-9}{7}$. Hence the gradient of Middle Way is $\frac{7}{9}$. The equation of L is $y-11=\frac{7}{9}(x-9)$, giving $y=\frac{7}{9}x+4$.

b. L crosses North Avenue at (0, 4).

$\frac{7}{9}x+4=0\Rightarrow x=-\frac{36}{7}$

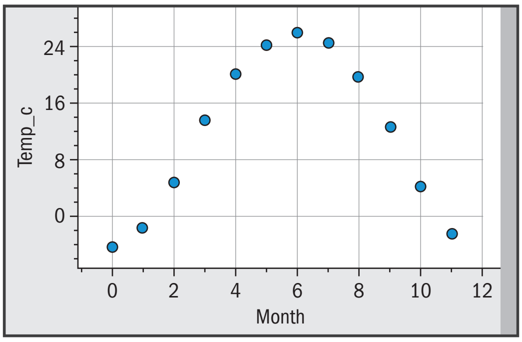
So L crosses East Way at $\left(\frac{-36}{7},0\right)$.

c. (HF) has equation $y-2=\frac{-9}{7}(x-16)$, giving $y=\frac{-9}{7}x+\frac{158}{7}$. (HF) crosses North Avenue at $\left(0,\frac{158}{7}\right)$. $\frac{-9}{7}x+\frac{158}{7}=0\Rightarrow x=\frac{158}{9}$ so (HF) crosses East Way at $\left(\frac{158}{9},0\right)$.

d. The distance from P to (0, 4) is 4 km. The distance from (0, 4) to M is $\sqrt{(0-9)^2+(4-11)^2}=\sqrt{130}$ and the distance from M to F is $\sqrt{(9-2)^2+(11-20)^2}=\sqrt{130}$. Hence the distance from P to F along North Way, then Middle Way, then (HF) is $4+2\sqrt{130}=26.8$ km.

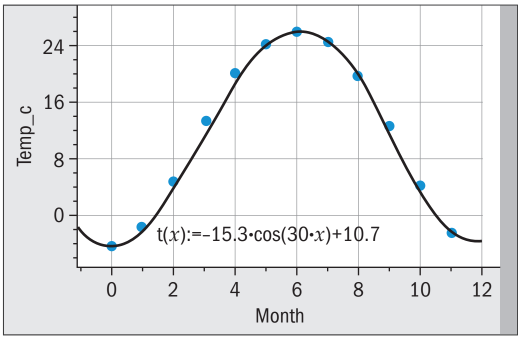
Another route is from P to $\left(0,\dfrac{158}{7}\right)$ which has distance $\dfrac{158}{7}$ along North Avenue, then from $\left(0,\dfrac{158}{7}\right)$ to the fire station, which has distance $\sqrt{(2-0)^2+\left(20-\dfrac{158}{7}\right)^2}$. Hence this route has length $\dfrac{158}{7}+\sqrt{(2-0)^2+\left(20-\dfrac{158}{7}\right)^2}\approx 25.8$ km. This is the shortest route.

18. a.



b. $p=\dfrac{360}{12}=30$, $w=\dfrac{26+(-4.6)}{2}=10.7$, $|k|=\dfrac{26-(-4.6)}{2}=15.3$ hence $k=-15.3$

c.



The model fits well at the turning points, and not so well in April and October, where the temperature is changing at a faster rate than the model.

Topic 2 SL Paper 2

1. a. The exponential model is more appropriate because it has a higher r^2 value and because a linear model would eventually predict a negative number of new subscribers whereas the minimum number of new subscribers is zero.

b. $A=339\,322$ and $B=0.969\,584$

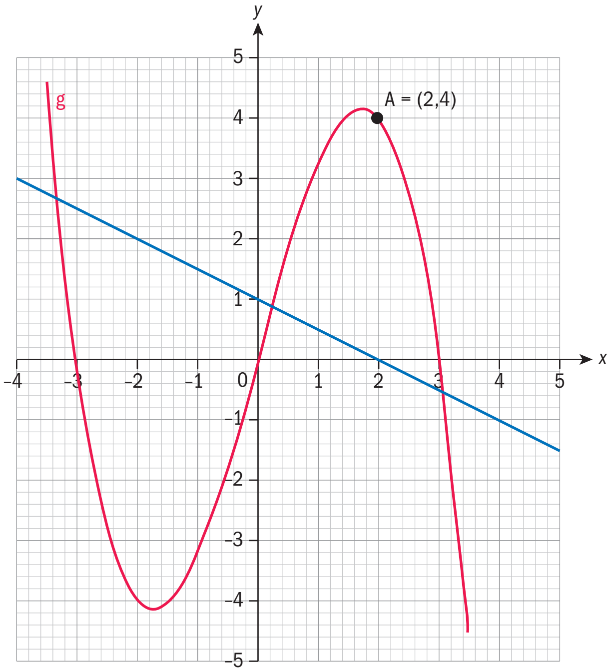
c. 249 000

d. After 62 weeks the number of new subscribers first falls below 50 000.

2. a. The domain of g is $-3.5 \le x \le 3.5$

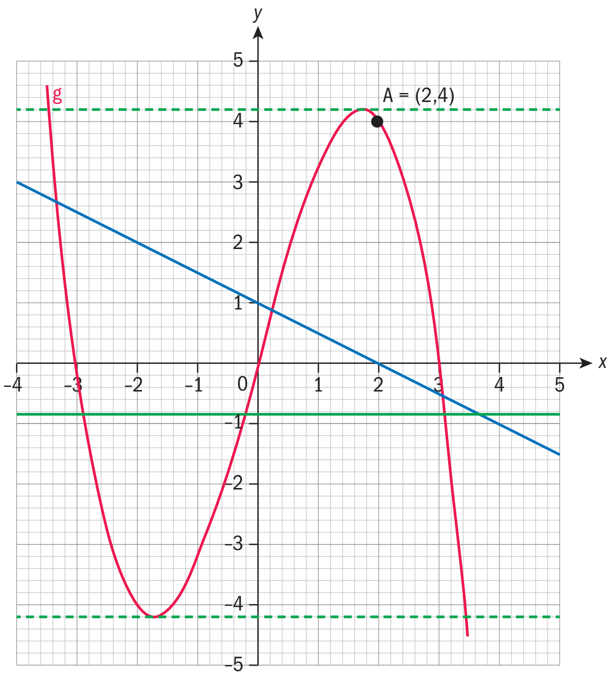
b. $g(2) = 4$

c.



d. $x = -3.4$, 0.2 or 3.0

e.

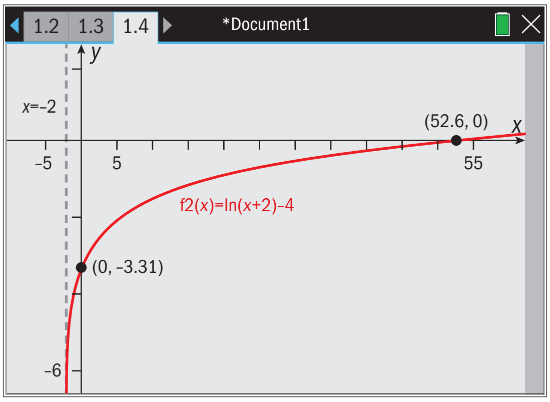


The solid green line shown gives three solutions. However if it was a horizontal tangent to the curve at one of the turning points (shown as green dashed lines), there would be two solutions to $g(x) = k$. Hence there are two values of k which gives two solutions to $g(x) = k$ since there are two turning points.

TOPIC 2 HL WORKED SOLUTIONS

Topic 2 HL Paper 1, Group 1

1. a.



b. The largest possible domain of f is $x > -2$.

c. $y = \ln(x+2) - 4 \Rightarrow \ln(x+2) = y + 4 \Rightarrow x + 2 = e^{y+4}$.
Therefore $f^{-1}(x) = e^{x+4} - 2$.

d. The domain of f^{-1} is $\mathbb{R}$ and the range is $y > -2$.

2. a. $g \circ f = 2\cos(3x+2) - 1$

b. The period of $g \circ f$ is $\dfrac{2\pi}{3}$

c. $h \circ t = 2a\cos(x) - a + b$ hence $a = \dfrac{5}{2}$ and $b = \dfrac{-3}{2}$.

d. $(h \circ t)(x) = 5\cos(x) - 4$ so the range is $[-9, 1]$

3. a. For the function $f(x) = \begin{cases} 8-(x-a)^2 & x<5 \\ x+2 & x \geq 5 \end{cases}$ to be continuous,

$8-(5-a)^2 = 5+2$. This gives $(5-a)^2 = 1$ hence $a = 4$.

b. The graph is moved 2 units to the right so the area remains at $36.2\ \text{m}^2$.

c. The graph is translated 3 units up so the area is $(36.2 + 3(p-q))\ \text{m}^2$.

d. The graph is stretched vertically by a factor of 2 and horizontally by a factor of 2. Hence the area is now $(4 \times 36.2)\ \text{m}^2 = 144.8\ \text{m}^2$.

4. a. $C = ae^{kt} + 18 \Rightarrow C - 18 = ae^{kt}$

Hence, $\ln(C-18) = \ln(ae^{kt}) = \ln(a) + \ln(e^{kt}) = \ln(a) + kt$.

b. The equation of the regression line of $\ln(C-18)$ on t is

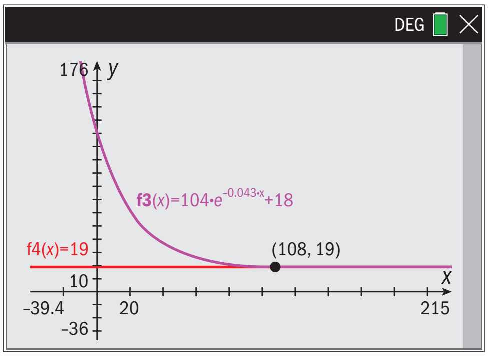
$\ln(C-18) = -0.0430t + 4.64$

c. Hence $C - 18 = e^{-0.0430t+4.64}$ therefore

$C = e^{-0.0430t\ +\ 4.64} + 18 = 104e^{-0.0430t} + 18$

so the value of k is –0.040 and the value of a is 104.

d. Sarah will have to wait 108 minutes in order for the cake to cool to 19 °C.

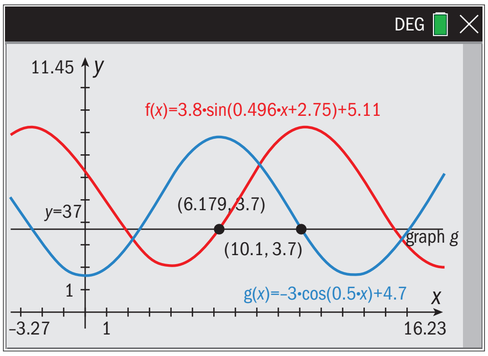


5. a. Sinusoidal regression with a GDC gave

$y = a\sin(b(t+c))+d = 3.08\sin(0.496x+2.75)+5.11.$

Hence the required form is $f(t)=3.08\sin(0.496(x+5.54))+5.11.$

b.



The GDC shows that both depths are above 3.7 m for 10.1 – 6.179 = 3.92 hours every morning.

6. a. $g^{-1}(x)=\sqrt[3]{x-7},\ f^{-1}(x)=5x+15$

b. $(f\circ g)(x)=f(x^3+7)=0.2(x^3+7)-3=0.2x^3-1.6$

$y=0.2x^3-1.6 \Rightarrow x=\sqrt[3]{5(y+1.6)}$ so $(f\circ g)^{-1}(x)=\sqrt[3]{5x+8}$

c. $(g^{-1}\circ f^{-1})(x)=g^{-1}(5x+15)=\sqrt[3]{5x+15-7}=\sqrt[3]{5x+8}$

d. $((f\circ g)\circ h)(x)=(f\circ g)(\ln(x))=0.2(\ln(x))^3-1.6$

$(f\circ(g\circ h))(x)=f(g(\ln x))=f((\ln(x))^3+7)=0.2((\ln(x))^3+7)-3$

$=0.2(\ln(x))^3+1.4-3=0.2(\ln(x))^3-1.6$

Topic 2 HL Paper 1, Group 2

7. a. As $x\to\infty$, $e^{-kx}\to 0$ since $k>0$. Hence $f(x)\to\dfrac{L}{1+C(0)}\to L$ so $y=L$ is a horizontal asymptote.

b. $f'(x)=-L(1+Ce^{-kx})^{-2}\times -kCe^{-kx}=\dfrac{kCL}{e^{kx}(1+Ce^{-kx})^2}.$

Since $L,k,C>0$ this has no turning point because $f'(x)$ can never be zero.

c. $f'(x) = kCL\, e^{-kx}\left(1+Ce^{-kx}\right)^{-2}$ so $f''(x)$ can be found with the product rule:

$$f''(x) = -k^2CL\, e^{-kx}\left(1+Ce^{-kx}\right)^{-2} + 2k^2C^2L\, e^{-2kx}\left(1+Ce^{-kx}\right)^{-3}$$

$$= -k^2CL\, e^{-kx}\left(1+Ce^{-kx}\right)^{-3}\left(1+Ce^{-kx}-2Ce^{-kx}\right)$$

$$= \frac{-k^2CL\left(1-Ce^{-kx}\right)}{e^{kx}\left(1+Ce^{-kx}\right)^3}.$$ $f''(x)$ changes sign when $1-Ce^{-kx}=0 \Rightarrow x = \frac{\ln C}{k}$

d. $f\left(\frac{\ln C}{k}\right) = \frac{L}{1+C\left(e^{-k\frac{\ln C}{k}}\right)} = \frac{L}{1+C\left(e^{-\ln C}\right)} = \frac{L}{2}$

8. **a.** T: A vertical stretch scale factor 2 followed by a vertical translation of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

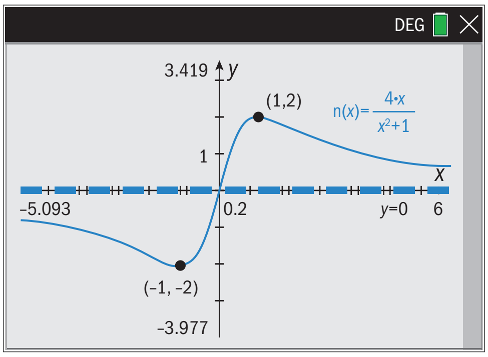
b. (3, 19)

c. In any order: S is a vertical stretch scale factor 2.1, a reflection in the x-axis and a horizontal stretch of scale factor $\frac{1}{3}$

d. $\left(\frac{-\pi}{9}, \frac{2.1\sqrt{3}}{2}\right)$

9. **a.** The turning points are (1,2) and (–1,–2) and the horizontal asymptote has equation $y = 0$.

b. Your sketch should show these features with the labels given, as well as the asymptotic behaviour.



c. The smallest domain $x \geq a$ on which $n^{-1}(x)$ exists is $x \geq 2$ since if $a < 2$, then $n(x)$ would not be one to one.

d. $x = \frac{4y}{y^2+1} \Rightarrow xy^2 + x = 4y$

Rearrange this equation to form a quadratic in y:

$$xy^2 - 4y + x = 0 \Rightarrow y = \frac{4 \pm \sqrt{16-4x^2}}{2x} = \frac{2 \pm \sqrt{4-x^2}}{x}$$

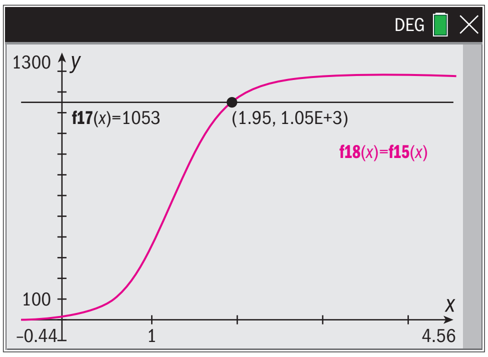
Since the $n(x)$ is in the first quadrant, $n^{-1}(x) = \frac{2+\sqrt{4-x^2}}{x}$

The domain of $n^{-1}(x)$ is $0 < x \leq 2$ and the range is $y \geq 1$.

10. **a.** $N_1(t) = \dfrac{1170}{1+76.8e^{-3.36t}}$

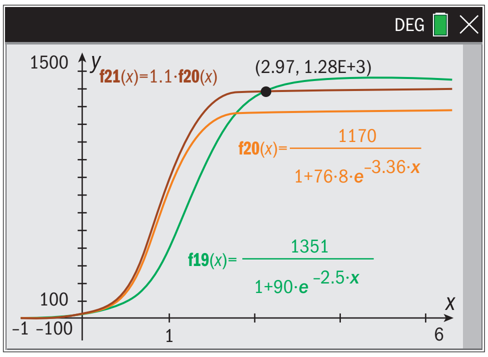
b. 1170

c. 90% of 1170 is 1053

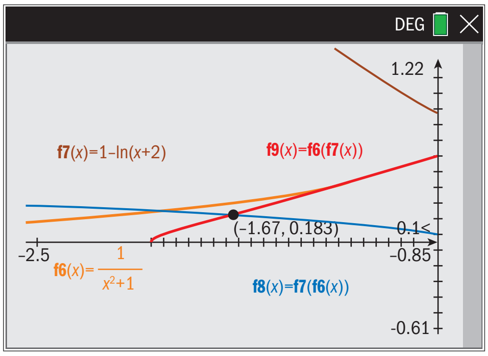


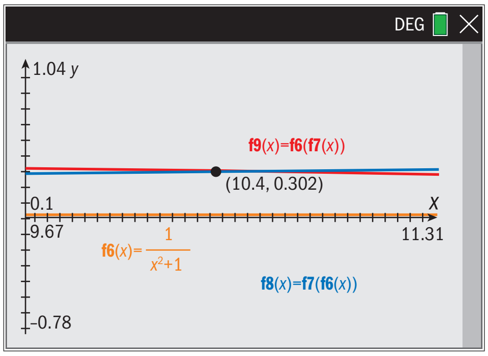
The population will reach 90% of its maximum possible after 1.95 hours.

d. $N_2 > 1.1N_1$ when $t > 2.97$.



11.





The two solutions are $x = -1.67$ and $x = 10.4$

12. $E(t) = 107 + 0.15t \Rightarrow t = \dfrac{E-107}{0.15}$

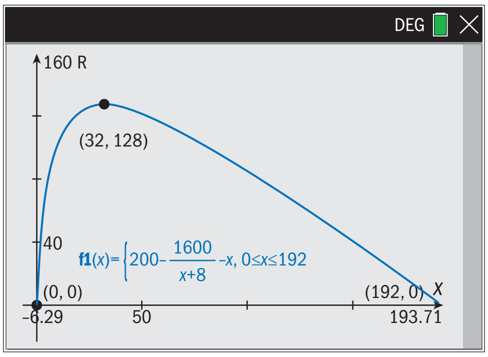
Hence,

$$L(t) = 10 + 1.2\times10^{-5}t = 10 + 1.2\times10^{-5}\left(\frac{E-107}{0.15}\right) = 10 + 8\times10^{-5}(E-107)$$

$$= 9.9914 + 8\times10^{-5}E$$

$$L(E) = 9.9914 + 8\times10^{-5}E$$

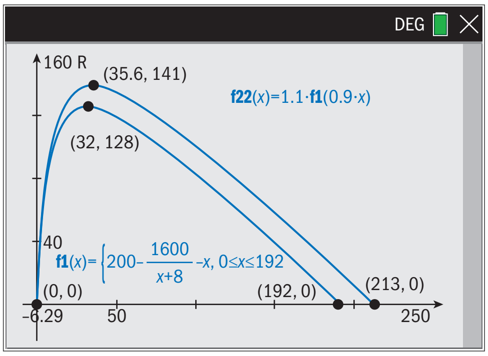
13. a.



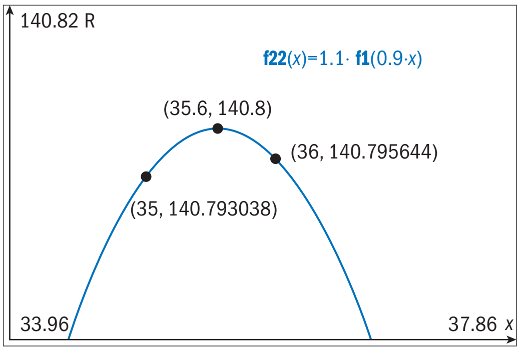
b. $a = 192$

c. A vertical stretch scale factor 1.1 and a horizontal stretch of scale factor $\frac{1}{0.9}$

d.



e. The new business model predicts that the maximum revenue can be increased from €128 000 to €140 796 by producing and selling 36 units.



14. a. Average speed $= \dfrac{\text{total distance}}{\text{total time}}$ and time $= \dfrac{\text{distance}}{\text{speed}}$

Chaya's total distance is 3 + 0 + 8 and

her total time is $\dfrac{3}{v}+\dfrac{1}{6}+\dfrac{8}{20}=\dfrac{180+10v+24v}{60v}=\dfrac{34v+180}{60v}=\dfrac{17v+90}{30v}$

Hence Chaya's average speed is

$$C(v)=\frac{11}{\frac{17v+90}{30v}}=\frac{330v}{17v+90}$$

b. $C=\dfrac{330v}{17v+90}\Rightarrow C(17v+90)=330v\Rightarrow 17Cv+90C=330v$

$\Rightarrow v(330-17C)=90C\Rightarrow v=\dfrac{90C}{330-17C}$ Hence $C^{-1}(v)=\dfrac{90v}{330-17v}$

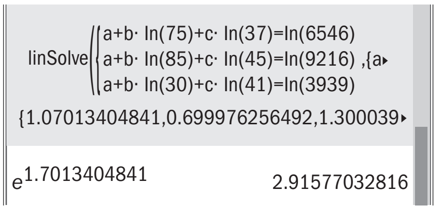
c. $C^{-1}(15) = \dfrac{90(15)}{330 - 17(15)} = 18\ \text{km h}^{-1}$

d. The claim is incorrect. $C(v)$ has a horizontal asymptote $C = \dfrac{330}{17} \approx 19.4\ \text{km h}^{-1}$.

No matter how fast Chaya cycles, the average speed can't exceed this value.

15. The system $\begin{cases} 6546 = A75^{b}37^{c} \\ 9216 = A85^{b}45^{c} \\ 3939 = A30^{b}41^{c} \end{cases}$ can be written as $\begin{cases} \ln(6546) = \ln(A) + b\ln(75) + c\ \ln(37) \\ \ln(9216) = \ln(A) + b\ln(85) + c\ \ln(45) \\ \ln(3939) = \ln(A) + b\ln(30) + c\ \ln(41) \end{cases}$

Use technology to solve the system for $\ln(A)$, b and c:



Hence, $A = 2.92,\ \ b = 0.700$ and $c = 1.30$

16. a. $a = 4,\ b = 3,\ a = 4,\ c = 2,\ d = 5$ and $e = 9$.

b. (11, 2)

c. $f(x) = \begin{cases} 4 - x & 1 \le x < 3 \\ (x-2)^2 & 3 \le x < 5 \\ 9 & 5 \le x < 8 \end{cases}$ is transformed to

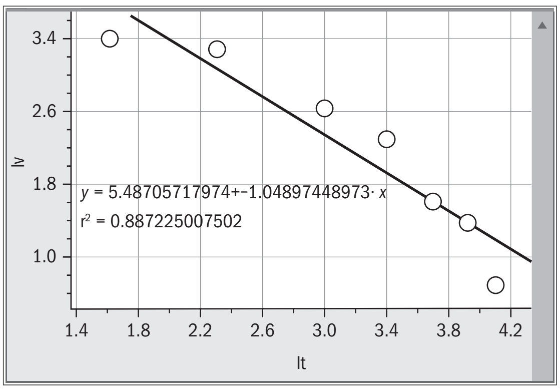
$$g(x) = \begin{cases} 15 - 2x & 3.5 \le x < 5.5 \\ 2(x - 4..5)^2 + 2 & 5.5 \le x < 7.5 \\ 20 & 7.5 \le x < 10.5 \end{cases}$$

17. a. $r^2 = 0.989\,887\,204\,673$

The slope is negative, hence

$r = -\sqrt{0.989\,887\,204\,673} = -0.994\,930\,753\,707\,57$

b.



c. $r^2 = 0.887\,225\,007\,502$.

The slope is negative, hence $r = -\sqrt{0.887\,225\,007\,502} = -0.941\,926\,221\,899\,57$

d. $V = ae^{ct}$ is the best model, since it is found from the semi-log graph given, which has a stronger correlation than the log-log data.

e. $V = ae^{ct} \Rightarrow \ln(V) = \ln(a) + ct$. Using the information given, $\ln(a) = 3.696\ 122...$

$\Rightarrow a = 40.3$ and $c = -0.049\,047\,9...$ Hence the model for the Tony's original data is $V = 40.3e^{-0.0490t}$

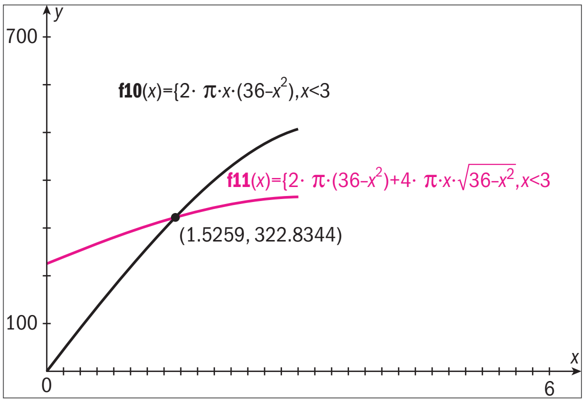
18. **a.** $[OD]$ is the radius of the sphere so $OD = 6$. $OB = x$ so $BD = \sqrt{36 - x^2}$, which is the radius of the cylinder. The height of the cylinder is $2x$ so the volume is $V(x) = \pi\left(\sqrt{36 - x^2}\right)^2(2x) = 2\pi x\left(36 - x^2\right)$

b. The domain of $V(x)$ is $0 \le x \le 3$

c. The surface area S is modelled by

$S(x) = 2\pi\left(\sqrt{36 - x^2}\right)^2 + 2\pi\sqrt{36 - x^2}\,(2x) = 2\pi\left(36 - x^2\right) + 4\pi x\sqrt{36 - x^2}$

d.

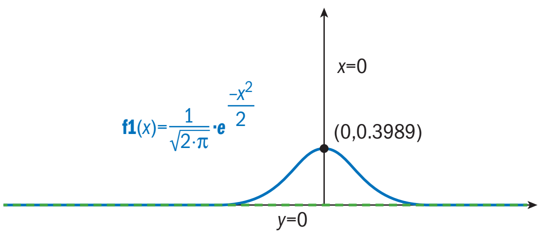


The height of the "perfect" cylinder is 3.05 cm and the radius 5.80 cm.

Topic 2 HL Paper 2

1. **a.** $x \in \mathbb{R}$

b.



c. $(-1, 0.242)$

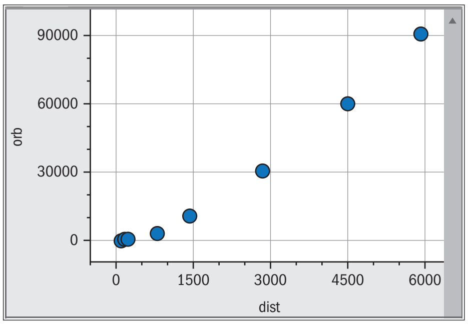
d. $(15, 0.3989)$

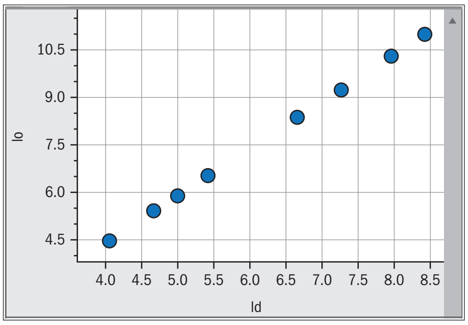
e. 1 unit squared

f. $p = q = 0.5$

g. The graph of g has been stretched horizontally by a factor of 2, and also stretched vertically by a factor of $\frac{1}{2}$, which preserves the area as 1 unit squared.

2. a.





Because of the large spread of magnitudes of both the independent and the dependent variables, the data is hard to represent on normal axes since so many data points are clustered near the origin. The log-log graphs enable us to represent the data on a more manageble scale.

b. $t = ad^n \Rightarrow \ln(t) = \ln(a) + n\ \ln(d)$

Using technology, $\ln(t) = -1.6178\ldots + (1.50137...)\ \ln(d)$

Hence $t = 0.193d^{1.50}$

c. 151 247 days, assuming that the model can be applied outside the data set given.

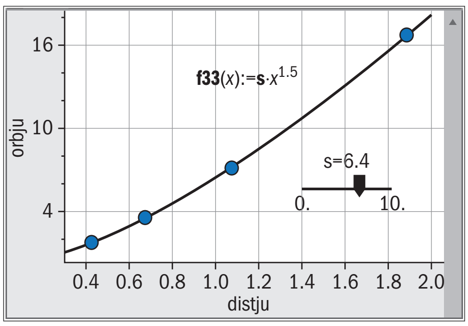
d. $t = (0.193)413^{1.50} = 1619.8782958151\ldots$ days

e. Percentage error $= \left|\dfrac{1619.8782958151 - 1682}{1682}\right| \times 100\% \approx 3.69\%$. Since the independent variable is 413, we are interpolating within the data set and this is a more valid process than extrapolation.

f. First, convert the units to the same as the planet data

moon	X (Average distance from Jupiter, millions of km)	Y (Time of one orbit, days)
Io	0.422	1.77083333
Europa	0.671	3.55
Ganymede	1.072	7.15416667
Callisto	1.883	16.6875

The graph displays a similar shape to the planetary data, so a vertical stretch was explored:



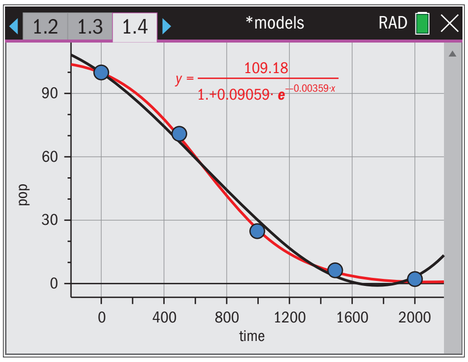
The model fits well when the coefficient is changed from 0.193 to 6.4.

Topic 2 HL Paper 3

a. i. Cubic model: $B_1 = 2.13\times10^{-8}t^3 - 0.00004t^2 - 0.05033t + 100.8$

ii. Logistic model: $B_1 = \dfrac{109.18}{1+0.09059e^{0.00359t}}$

b. i.

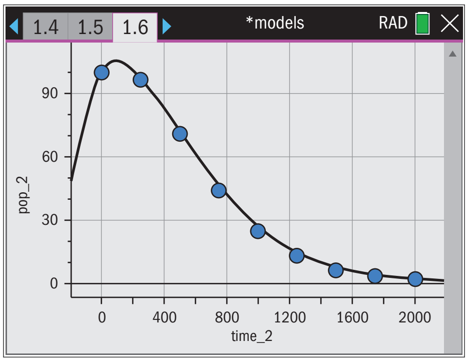


ii. The logistic function fits the points better than the cubic. Also, the cubic model implies that the population of bacteria starts to grow again.

iii. The limiting value of the logistic function is zero. However, the cubic model – when used to extrapolate from this data set – would imply the bacteria grow to infinity.

c. The model B_2 should have a maximum value near the start of the experiment.

d.



The modified model B_3 shows the shape required.

e. $B_3(100) = 105.83$

Percentage error $= \left|\dfrac{105.83-105}{105}\right| \times 100\% = 0.794\%$

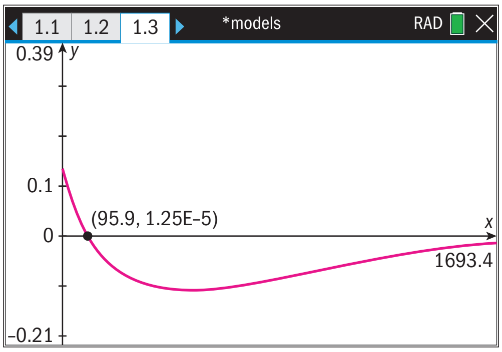
f. $B_3 = 600\left(4+e^{-0.011t}+e^{0.0029t}\right)^{-1}$

$$\Rightarrow \frac{dB_3}{dt} = -600\left(4+e^{-0.011t}+e^{0.0029t}\right)^{-2}\times\left(-0.011e^{-0.011t}+0.0029e^{0.0029t}\right)$$

g. $\dfrac{dB_3}{dt} = 0 \Rightarrow -0.011e^{-0.011t}+0.0029e^{0.0029t} = 0$

$$0.011e^{-0.011t} = 0.0029e^{0.0029t} \Rightarrow e^{0.0139t} = \frac{0.011}{0.0029} \Rightarrow t = \frac{\ln\left(\frac{0.011}{0.0029}\right)}{0.0139} = 95.9$$

This value is confirmed by the GDC since this graph of $y = \frac{dB_3}{dt}$ shows that $\frac{dB_3}{dt}$ changes sign from positive to negative at $t = 95.9$



h. $y = p\left(q + e^{rt} + e^{st}\right)^{-1}$

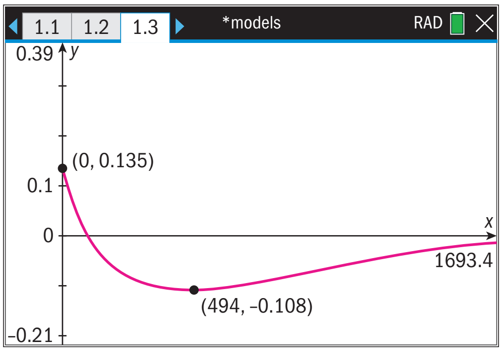
$\Rightarrow \frac{dy}{dt} = -p\left(q + e^{rt} + e^{st}\right)^{-2} \times \left(re^{rt} + se^{st}\right)$

$\frac{dy}{dt} = 0 \Rightarrow re^{rt} + se^{st} = 0$

$\Rightarrow re^{rt} = -se^{st} \Rightarrow e^{(s-r)t} = \frac{-r}{s} \Rightarrow (s-r)t = \ln\left(\frac{-r}{s}\right)$

The parameters r and s determine the time at which the population reaches its maximum since the derivative is zero at $t = \frac{\ln\left(\frac{-r}{s}\right)}{s-r}$, which depends only on these parameters.

i. The graph of $y = \frac{dB_3}{dt}$ shows that the global maximum value of the first derivative is 0.135 and the global minimum value is –0.108. Since these values are not equal in size, the researcher was incorrect. However, the rates are similar.



TOPIC 3 SL WORKED SOLUTIONS

Topic 3 SL Paper 1, Group 1

1. a. Area of triangle $= \frac{1}{2} \times 3 \times 3 \times \sin 140 = 2.892... \approx 2.89 \text{ cm}^2$

b. Area of sector $= \frac{40}{360} \times \pi \times 3^2 = 3.14159... \approx 3.14 \text{ cm}^2$

2. Distance $= \sqrt{(108.3 - 102.8)^2 + (42.2 - 39.1)^2 + (4.5 - 2.9)^2}$

$= \sqrt{42.42} \approx 6.51 \text{ km}$

3. a. Volume $= \pi \times 8^2 \times 10 = 2010.6... \approx 2011 \text{ cm}^3$

b. Volume $= \frac{35}{360} \times 2010.6... = 195.47... \approx 195 \text{ cm}^3$

4. a. Midpoint of $[AB]$ is $\left(\frac{2+(-1)}{2}, \frac{8+10}{2}\right) = (0.5, 9)$

b. Gradient of line joining A and $B = \frac{10-8}{-1-2} = -\frac{2}{3}$

Gradient of $l = \frac{3}{2} = 1.5$

Equation of l is $y - 9 = 1.5(x - 0.5)$ or $y = 1.5x + 8.25$

5. a. $\tan 48° = \frac{CD}{250}$

$CD = 277.653... = 278$ m (3 sf)

b. Angle of depression from A is equal to the angle of elevation from B.

$\tan A\hat{B}C = \frac{\frac{4}{3} \times 277.653...}{250}$

or $90 - \tan^{-1}\left(\frac{250}{\frac{4}{3} \times 277.653...}\right)$

gives angle of depression $= 56.0°$

6. a. $\frac{BC}{\sin 34°} = \frac{5}{\sin 120°}$

$BC = 3.22850... = 3.23$ cm (3 sf)

b. Area $= \frac{1}{2}(5)(3.22850...)\sin 26° = 3.54 \text{ cm}^2$

Topic 3 SL Paper 1, Group 2

7. **a.** $y = 5,\ x = 4$ and $y = -\frac{3}{2}x + 11$

b. Using the formula for the area of a trapezoid and rectangle:

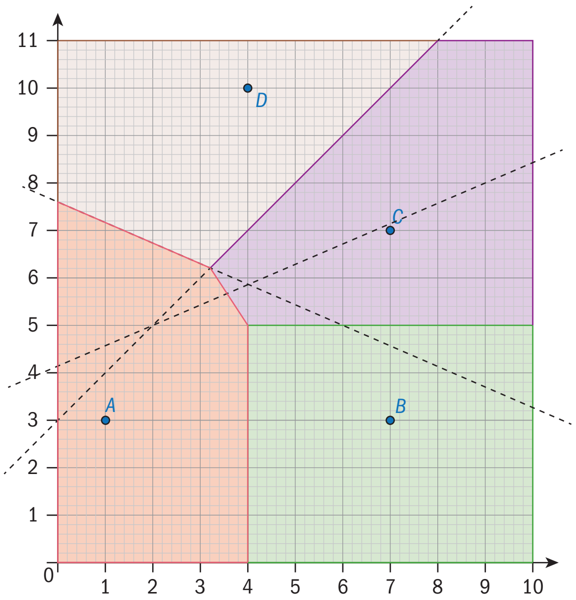
i. A: $\frac{1}{2} \times 4 \times (11 + 5) = 32\ \text{km}^2$

ii. B: $6 \times 5 = 30\ \text{km}^2$

iii. C: $\frac{1}{2} \times 6 \times (10 + 6) = 48\ \text{km}^2$

The third area can also be found by subtracting the first two found from 110.

c.



8. Midpoint of the tunnel is at $\left(\frac{10.1 + 8.5}{2}, \frac{3.2 + 1.7}{2}, \frac{-0.8 - 0.9}{2}\right) = (9.3, 2.45, -0.85)$

Distance from C to this point is $\sqrt{(9.0 - 9.3)^2 + (2.0 - 2.45)^2 + (0 + 0.85)^2}$

≈ 1.01 km

9. **a.** $BC = \sqrt{(250 + 120)^2 + (120 - 230)^2 + (75 - 140)^2} = 391.4\ldots \approx 391$ m

b. $AC = \sqrt{120^2 + 230^2 + 140^2} = 294.7\ldots \approx 295$ m

Area $= \frac{1}{2} \times 391 \times 295 \times \sin 47° = 42\,196 \approx 42\,200\ \text{m}^2$

10. **a.** $20 = 2r + \frac{\theta}{360} 2\pi r \Rightarrow \theta = \frac{180(20 - 2r)}{\pi r}$

b. $A = \frac{\theta}{360} \times \pi r^2$

$= \frac{1}{360} \times \frac{180(20 - 2r)}{\pi r} \times \pi r^2 = \frac{(20 - 2r)r}{2}$

$= (10 - r)r = 10r - r^2$

c. Plot $A = 10r - r^2$ on the GDC or use the symmetry of a quadratic to obtain the value $r = 5$ cm.

11. a. $100 = \frac{1}{3}\pi r^2(8)$

$r = 3.45$ (cm) (3.45494… (cm))

b. $l^2 = 8^2 + (3.45494\ldots)^2$

$l = 8.71$ (cm) (8.71416… (cm))

c. $\pi \times 3.45494\ldots \times 8.71416\ldots$

$= 94.6\text{cm}^2$ ($94.5836\ldots\text{cm}^2$)

12. a. $AC^2 = 8^2 + 6^2$

$AC = 10$

$VM^2 = 13^2 - 5^2$

$VM = 12$ (cm)

b. $\frac{1}{3} \times 8 \times 6 \times 12$

$= 192\,\text{cm}^3$

Topic 3 SL Paper 1, Group 3

13. a. $AB^2 = 6^2 + 8^2 - 2 \times 6 \times 8 \cos 60 = 52$

$AB = \sqrt{52} = 7.2111\ldots \approx 7.21$ m

b. $\cos\left(B\hat{O}A\right) = \frac{4^2 + 4^2 - 52}{2 \times 4 \times 4} = -0.625$

$\Rightarrow B\hat{O}A = 128.68\ldots \approx 129°$

Arc length $AB = \frac{128.68\ldots}{360} \times 2\pi \times 4 = 8.9837\ldots \approx 8.98$ m

Amount of edging needed $= 8.98 + 7.21 \approx 16.2$ m

14. a. i. $OC = r\cos 60 = 0.5r$

ii. Area of triangle $= \frac{1}{2} \times r \times 0.5r \times \sin 60 = 0.2165\ldots r^2$

b. Area of sector $= \frac{60}{360} \times \pi \times r^2 = \frac{\pi}{6} r^2$ $\left(\text{or } 0.5235\ldots r^2\right)$

Hence $0.5235\ldots r^2 - 0.2165\ldots r^2 = 50$

$0.307\ldots r^2 = 50 \Rightarrow r^2 = 162.8\ldots$

$r = 12.759\ldots \approx 12.8$ cm

15. a. i. 15 m

ii. $\frac{1}{3} \times 8^2 \times 15 = 320\,\text{m}^3$

b. $a = 4$

c. The light is half way between the midpoint of [AB] and the vertex.

Midpoint of [AB] has coordinates $(4, 0, 0)$.

Coordinates of light $= \left(\frac{4+4}{2}, \frac{4+0}{2}, \frac{15+0}{2}\right) = (4, 2, 7.5)$

d. Height of face is the distance between V and $(4,0,0)$

$$=\sqrt{(4-4)^2+(4-0)^2+(15-0)^2}=\sqrt{241}=15.52\ldots\approx 15.5\text{ m}$$

Surface area of pyramid $=\left(4\times\frac{1}{2}\times 8\times 15.5\right)=248.38\ldots\approx 248\text{ m}^3$

16. Attempt to use tan, or sine rule, in triangle BXN or BXS

$$NX=80\tan 55°\left(=\frac{80}{\tan 35°}\right)=114.25$$

$$SX=80\tan 65°\left(=\frac{80}{\tan 25°}\right)=171.56$$

Attempt to use cosine rule

$SN^2 = 171.56^2 + 114.25^2 - 2 \times 171.56 \times 114.25 \cos 70°$

$SN = 171\text{ m}$

17. a. 3.2

b. $a=\frac{1}{2}\times 4\times 6=12\text{ km}^2$

c. $a_C=\frac{1}{2}\times 4\times 2=4\text{ km}^2$

By symmetry $a_A=a_B=\frac{12-4}{2}=4\text{ km}^2$

$$P_D=\frac{6.5\times 4+7.2\times 4+3.2\times 4}{12}=5.633\ldots\approx 5.63$$

18. a. Volume of water $=\pi\times 8^2\times 12$

$=768\pi$

$=2412.74\ldots=2410\text{ cm}^3$ (3 sf).

b. $\frac{4}{3}\pi\times 2.9^3+768\pi=\pi\times 8^2 h$ or $\frac{4}{3}\pi\times 2.9^3=\pi\times 8^2(h-12)$

gives $h=12.5081\ldots=12.5$ cm (3 sf)

Topic 3 SL Paper 2

1. a. i. Midpoint is (6, 3).

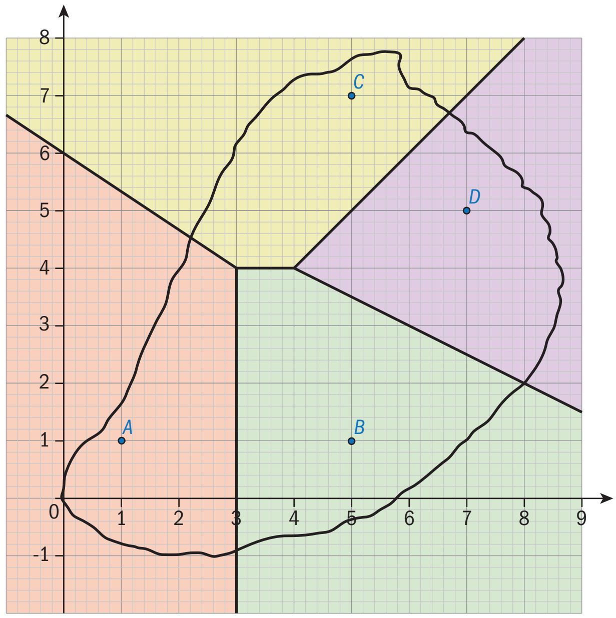
ii. Gradient is 2.

b. Either $y-3=-\frac{1}{2}(x-6)\Rightarrow y=-\frac{1}{2}x+6$

or $y=-\frac{1}{2}x+c\Rightarrow 3=-\frac{1}{2}\times 6+c\Rightarrow c=6$

c. $y=x$

d.



e. (3,4)

f. $\sqrt{13} \approx 3.61$ km from A, B and C and $\sqrt{17} \approx 4.12$ km from D

2. a. $BD^2 = 40^2 + 84^2$

$BD = 93.0376\ldots$

$= 93$

b. $\cos B\hat{C}D = \dfrac{115^2 + 60^2 - 93^2}{2 \times 115 \times 60}$

$= 53.7°\ (53.6679\ldots°)$

c. $\dfrac{1}{2}(40)(84) + \dfrac{1}{2}(115)(60)\sin(53.6679\ldots)$

$= 4460\ \text{m}^2\ (4459.30\ldots\ \text{m}^2)$

d. i. $\dfrac{(40+60)(84+115)}{4}$

$= 4980\ \text{m}^2\ (4975\ \text{m}^2)$

ii. $\left|\dfrac{4975 - 4459.30\ldots}{4459.30\ldots}\right| \times 100$

$= 11.6\ (\%)\ (11.5645\ldots)$

3. a. i. $222 = \dfrac{1}{2}x(x+3) + (x+3)(x+5)$

OR

$222 = (x+3)(2x+5) - 2\left(\dfrac{1}{4}\right)x(x+3)$

ii. $222 = \dfrac{1}{2}x^2 + \dfrac{3}{2}x + x^2 + 3x + 5x + 15$

$3x^2 + 19x - 414 = 0$

b. Solving the equation in (a. ii) by factorising or use of *GDC*

$x = 9\left(\text{and } x = -\frac{46}{3}\right)$

$CD = 12$ (cm)

c. Divide triangle *ABE* into 2 right-angled triangles.

The base of each triangle is $\frac{1}{2}(9+3) = 6$

$\tan\left(\frac{B\hat{A}E}{2}\right) = \frac{6}{9}$

$B\hat{A}E = 67.3801\ldots°$

$= 67.4°$

d. $2\sqrt{9^2+6^2}+12+2(14)$

$= 61.6$ (cm) (61.6333… (cm))

e. $F\hat{B}C = 90+\left(\frac{180-67.4}{2}\right)(=146.3°)$

OR

$180-\frac{67.4}{2}$

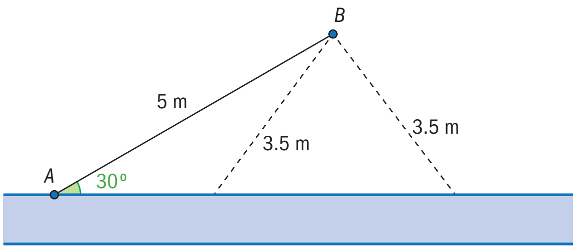
$CF^2 = 8^2 + 14^2 - 2(8)(14)\cos(146.3°)$

$CF = 21.1$ (cm) (21.1271…)

TOPIC 3 HL WORKED SOLUTIONS

Topic HL Paper 1, Group 1

1. a.



b. $\dfrac{\sin\theta}{5} = \dfrac{\sin 30°}{3.5} \Rightarrow \sin\theta = 0.714...$

$\theta = 45.58...$ or $180 - 45.58... = 134.4...$

For the smaller triangle the third angle is $180 - 30 - 134.4... = 15.58...$

$\text{Area} = \dfrac{1}{2} \times 5 \times 3.5 \times \sin(15.58...) \approx 2.35\ \text{m}^2$

2. a. $\begin{pmatrix} 2 & 1 \\ a & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} b \\ 3 \end{pmatrix}$

$8 + 3 = b \Rightarrow b = 11$

$4a - 3 = 3 \Rightarrow a = \dfrac{6}{4} = 1.5$

b. $\begin{pmatrix} 2 & 1 \\ 1.5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -7 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1.5 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ -7 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

Point is (–2, 4)

3. a. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 105 \\ 226 \\ 12 \end{pmatrix} + 0.5\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow a = -210,\ b = -452,\ c = -24$

b. $\text{Speed} = \sqrt{210^2 + 452^2 + 24^2} \approx 499$ or $500\ \text{km h}^{-1}$

4. a. i. $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

ii. $\overrightarrow{AC} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$

b. $\text{Area} = \dfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \left|\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}\right| = \dfrac{1}{2} \times \left|\begin{pmatrix} -8 \\ 1 \\ -2 \end{pmatrix}\right|$

$= \dfrac{1}{2}\sqrt{8^2 + 1^2 + 2^2} = \dfrac{1}{2} \times \sqrt{69} \approx 4.15$

5. a. $\dot{\boldsymbol{x}} = \begin{pmatrix} 3t^2 + c_1 \\ 2t + c_2 \end{pmatrix}$

At $t = 0$, $\dot{\boldsymbol{x}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow c_1 = 2,\ \ c_2 = 1$

Hence $\dot{\boldsymbol{x}} = \begin{pmatrix} 3t^2 + 2 \\ 2t + 1 \end{pmatrix}$

b. $\boldsymbol{x} = \begin{pmatrix} t^3 + 2t + c_3 \\ t^2 + t + c_4 \end{pmatrix}$

At $t = 0$, $\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow c_3 = 0,\ \ c_4 = 0$

Hence $\boldsymbol{x} = \begin{pmatrix} t^3 + 2t \\ t^2 + t \end{pmatrix}$

c. At $t = 2$ $\boldsymbol{x} = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$

Distance $= \sqrt{12^2 + 6^2} = \sqrt{180} \approx 13.4$ m

6. a. Prim's or Kruskal's algorithm

b. **EITHER**

using Prim's algorithm, starting at A

Edge	**Cost**
AC	4
CD	3
CF	4
FE	4
AB	5

lowest cost road system contains roads *AC*, *CD*, *CF*, *FE* and *AB* cost is 20

OR

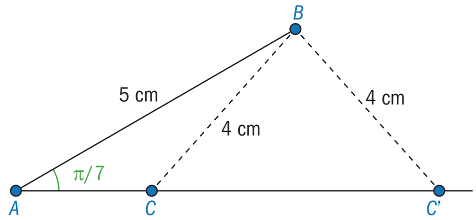
using Kruskal's algorithm

Edge	**Cost**
CD	3
CF	4
FE	4
AC	4
CB	5

lowest cost road system contains roads *CD*, *CF*, *FE*, *AC* and *AB* cost is 20

Topic 3 HL Paper 1, Group 2

7. a.



Let $AC = x$

$4^2 = x^2 + 5^2 - 2\times x\times 5\cos\left(\frac{\pi}{7}\right)$

$\Rightarrow x^2 - 10\cos\left(\frac{\pi}{7}\right)x + 9 = 0$

$\Rightarrow x = 1.144... \approx 1.14\text{ m} \text{ or } 7.865\ldots \approx 7.87\text{ cm}$

b. Area $= \frac{1}{2}\times 5\times 7.865...\times \sin\left(\frac{\pi}{7}\right) \approx 8.53\text{ cm}^2$

8. a. Area $= \frac{1}{2}(5-3)(6-2) = 4$

b. Area of image $= \det\begin{pmatrix} 2\sin\theta & -2\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}\times 4$

$= \left(2\sin^2\theta + 2\cos^2\theta\right)\times 4 = 8\left(\sin^2\theta + \cos^2\theta\right) = 8$

9. a. $A = \frac{1}{2}\times 10^2\times\theta - \frac{1}{2}\times 10^2\times\sin\theta = 50\theta - 50\sin\theta$

b. Unshaded area $= \frac{\pi\times 10^2}{2} - 50(\theta - \sin\theta) = 50(\pi - \theta + \sin\theta)$

Solve $50(\theta - \sin\theta) = \frac{1}{2}\times 50(\pi - \theta + \sin\theta)$

$\theta \approx 1.969$

10. $\sqrt{m^2 + n^2} = 1 \Rightarrow m^2 + n^2 = 1$

$\boldsymbol{u}\cdot\boldsymbol{v} = 0$

$\Rightarrow -3\times 0 + 1\times m + 1\times n = 0 \Rightarrow m + n = 0$

$\Rightarrow m = -n$

$(-n)^2 + n^2 = 1 \Rightarrow 2n^2 = 1 \Rightarrow n = \pm\sqrt{0.5} = \pm 0.70710\ldots$

Either $n = \sqrt{0.5}$, $m = -\sqrt{0.5}$ or $n = -\sqrt{0.5}$, $m = \sqrt{0.5}$

11. **a.** The graph has no Euler circuit because not all the vertices are even.

b. The four vertices of odd degree are A, D, F, J. These can be paired up as:

AD JF	$15 + 15 = 30$
AJ DF	$30 + 13 = 43$
AF JD	$22 + 15 = 37$

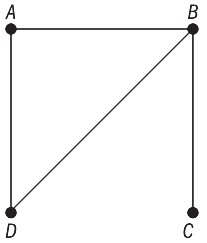
Least possible cost is $152 + 30 = 182$ USD

Routes to be repeated are AD, JI and IF

c. Start at A and finish at J or vice versa.

Extra cost is $DF = 13$ USD so would save $30 - 13 = 17$ USD

12. **a.**



b.

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 & 0 \end{pmatrix} \end{array}$$

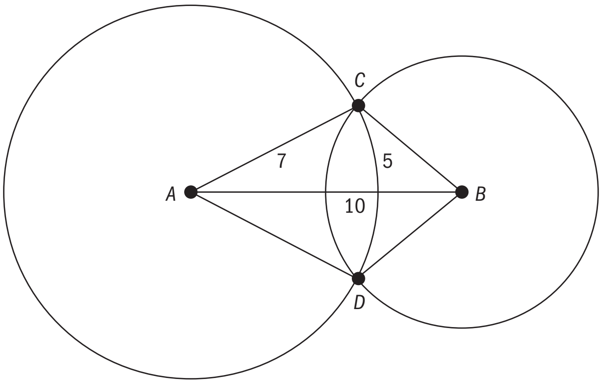
c.

$$\begin{pmatrix} 0.25 \\ 0.375 \\ 0.125 \\ 0.25 \end{pmatrix}$$

d. He spends most time in room B and least time in room C. We are assuming that when he goes into a room he spends an equal amount of time in each one.

Topic 3 HL Paper 1, Group 3

13.



Use of cosine rule:

$$C\hat{A}B = \arccos\left(\frac{49+100-25}{2\times 7\times 10}\right) = 0.48276\ldots (= 27.660\ldots°)$$

$$C\hat{B}A = \arccos\left(\frac{25+100-49}{2\times 5\times 10}\right) = 0.70748\ldots (= 40.535\ldots°)$$

Attempt to subtract triangle area from sector area:

$$\text{area} = \frac{1}{2}\times 49\left(2C\hat{A}B - \sin 2C\hat{A}B\right) + \frac{1}{2}\times 25\left(2C\hat{B}A - \sin 2C\hat{B}A\right)$$

$= 3.0579\ldots + 5.3385\ldots$

$= 8.85\,(\text{km}^2)$

14. a. i. $\begin{pmatrix} 1.1 & 0 \\ 0 & 1.1 \end{pmatrix}$

ii. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

b. $\mathbf{T} = \begin{pmatrix} 1.1 & 0 \\ 0 & 1.1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1.1 \\ 1.1 & 0 \end{pmatrix}$

i. $\mathbf{T}^n = \begin{pmatrix} 0 & 1.1^n \\ 1.1^n & 0 \end{pmatrix}$ when n is odd

ii. $\mathbf{T}^n = \begin{pmatrix} 1.1^n & 0 \\ 0 & 1.1^n \end{pmatrix}$ when n is even

c. $1.1^n > 20 \Rightarrow n \geq 32$

As 32 is even the coordinates are $(1.1^{32}, 0) \approx (21.1, 0)$

15. a. $\sqrt{5.2^2+3.9^2}=6.5$

b. Bearing = $\arctan\left(\dfrac{5.2}{3.9}\right)=053.1°$

c. Time = $\dfrac{12}{6.5}=1.846\ldots$ hours $\approx$ 1 hour, 51 minutes

d. Displacement from the port when $t=0$ is $-\dfrac{12}{6.5}\begin{pmatrix}5.2\\3.9\end{pmatrix}$

Displacement at time t is

$$\boldsymbol{r}=-\frac{12}{6.5}\begin{pmatrix}5.2\\3.9\end{pmatrix}+t\begin{pmatrix}5.2\\3.9\end{pmatrix}=\begin{pmatrix}-9.6\\7.2\end{pmatrix}+t\begin{pmatrix}5.2\\3.9\end{pmatrix}$$

16. a. $\begin{pmatrix}5\cos 30\\5\sin 30\end{pmatrix}\approx\begin{pmatrix}4.33\\2.5\end{pmatrix}$

b. Integrate acceleration to obtain velocity and velocity to obtain displacement. Use the initial conditions to find the values of the constant.

$$\boldsymbol{a}=\begin{pmatrix}0\\-9.8\end{pmatrix}\Rightarrow\boldsymbol{v}=\begin{pmatrix}4.33\\2.5-9.8t\end{pmatrix}$$

$$\boldsymbol{r}=\begin{pmatrix}4.33t\\1.5+2.5t-4.9t^2\end{pmatrix}$$

c. For student B's beanbag initial displacement is $\begin{pmatrix}20\\1.0\end{pmatrix}$ and the velocity is

$\begin{pmatrix}-6\cos\theta\\6\sin\theta\end{pmatrix}$ where θ is the acute angle made with the horizontal.

$$\boldsymbol{r}_B=\begin{pmatrix}20-(6\cos\theta)t\\1.0+(6\sin\theta)t-4.9t^2\end{pmatrix}$$

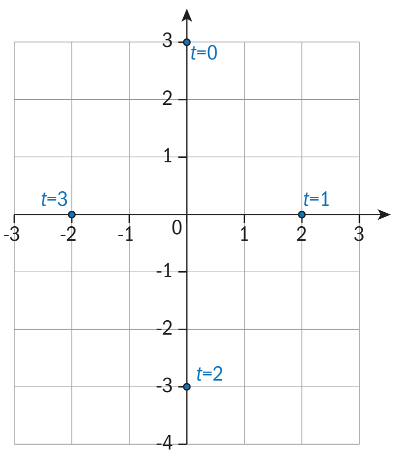
If they collide $20-(6\cos\theta)t=4.33\ldots t\Rightarrow t=\dfrac{20}{6\cos\theta-4.33\ldots}$

$$1.0+(6\sin\theta)t-4.9t^2=1.5+2.5t-4.9t^2\Rightarrow(6\sin\theta-2.5)t=0.5$$

Substitute in $(6\sin\theta-2.5)\dfrac{20}{6\cos\theta-4.33\ldots}=0.5$

Solve to find $\theta\approx 24.9°$

17. a.



b. $v = \begin{pmatrix} \pi\cos\left(\frac{\pi t}{2}\right) \\ -\frac{3\pi}{2}\sin\left(\frac{\pi t}{2}\right) \end{pmatrix}$

c. $|v| = \sqrt{\left(\pi\cos\left(\frac{\pi t}{2}\right)\right)^2 + \left(\frac{3\pi}{2}\sin\left(\frac{\pi t}{2}\right)\right)^2} = \frac{\pi}{2}\sqrt{4\cos^2\left(\frac{\pi t}{2}\right) + 9\sin^2\left(\frac{\pi t}{2}\right)}$

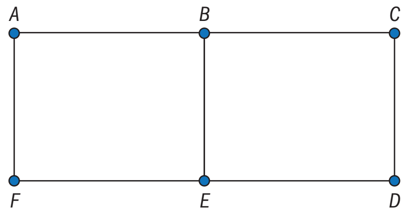
$= \frac{\pi}{2}\sqrt{4 + 5\sin^2\left(\frac{\pi t}{2}\right)}$ [using $\cos^2\left(\frac{\pi t}{2}\right) = 1 - \sin^2\left(\frac{\pi t}{2}\right)$]

d. Maximum when $\sin^2\left(\frac{\pi t}{2}\right) = 1 \Rightarrow \sin\left(\frac{\pi t}{2}\right) = \pm 1$

$\Rightarrow t = 1$ and $t = 3$

e. These are the two points when the planet is closest to the star.

18. a.



b. i. From the matrix raised to the power 6, the number of walks from A to A is 50

ii. 2 walks; it is only possible if no vertices are repeated

c. $ABCD$, $ABED$, $AFED$

d. $P(ABCD) = \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{12}$

$P(ABED) = \frac{1}{2} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{18}$

$P(AFED) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{12}$

Total probability $= \frac{1}{12} + \frac{1}{18} + \frac{1}{12} = \frac{2}{9}$

Topic 3 HL Paper 2

1. a. $(-3, 1, 2)$

b. i. $\overrightarrow{AB} = 5\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ -5 \end{pmatrix}$

ii. $\left|\overrightarrow{AB}\right| = \sqrt{10^2 + 10^2 + 5^2} = 15$

c. $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

d. $\boldsymbol{r} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + (t-3)\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

e. $\overline{AB} \bullet \overline{AC} = \begin{pmatrix} 10 \\ 10 \\ -5 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} = 90$

$\cos B\hat{A}C = \frac{90}{15\sqrt{56}} \left(= \frac{6}{\sqrt{56}} = \frac{3}{\sqrt{14}} \right) \approx 0.802$

Note: could have used $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

f. Let the distance between P_1 and P_2 be x.

Using the cosine rule $x^2 = 15^2 + \left(\sqrt{56}\right)^2 - 2 \times 15 \times \sqrt{56} \times \frac{6}{\sqrt{56}} = 101$

$\Rightarrow x = \sqrt{101} \approx 10.0$ m

2. a. Order of edge selection is AD, DB, BC

Weight of minimum spanning tree is 9 + 11 + 8 = 28 km

b.

	A	B	C	D	E
A		12	15	9	5
B	12		8	11	8
C	15	8		18	16
D	9	11	18		14
E	5	8	16	14	

Route EAC = 5 + 15 = 20, EBC = 8 + 8 = 16 so 16 km is the least distance.

Route EAD = 5 + 9 = 14, EBD = 8 + 11 = 19 so 14 km is the least distance.

c. $AEBCDA$; length = 5 + 8 + 8 + 18 + 9 = 48 km

d. Lower bound is 28 + 5 + 8 = 41 km

e. Weight of minimum spanning tree in the remaining graph = 29 km
Hence lower bound = 29 + 8 + 8 = 45 km

f. Let the length of the journey be x.

Then $45 \leq x \leq 48$

Topic 3 HL Paper 3

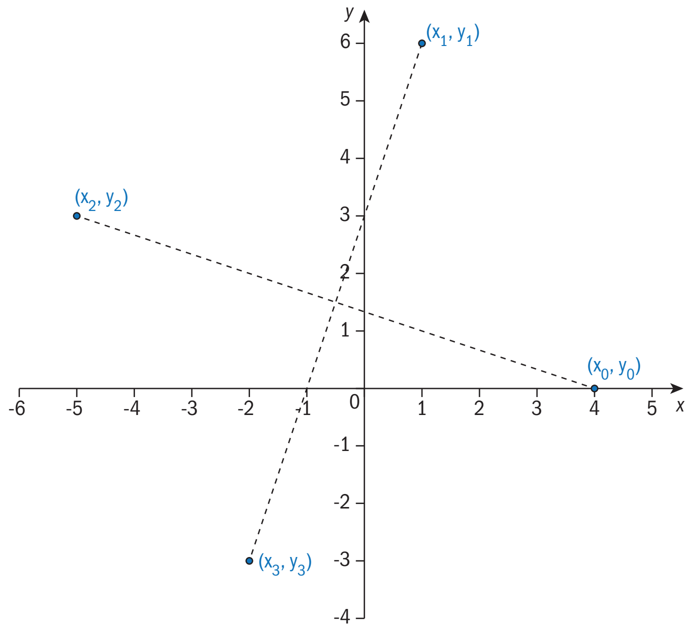
1. a. A rotation of 90° anticlockwise about $(0, 0)$

b. i. $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

ii. $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$

iii. $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -5 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

c.



d. Midpoint of $(4, 0)$ and $(-5, 3)$ is $\left(\frac{4-5}{2}, \frac{0+3}{2}\right) = (-0.5, 1.5)$

e. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -0.5 \\ 1.5 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1.5 \\ -0.5 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 1.5 \end{pmatrix}$

f. $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

g. $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 4 \\ -2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \end{pmatrix}$

A is $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} + \begin{pmatrix} -8 \\ 4 \end{pmatrix}$

h. $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} -8 \\ 4 \end{pmatrix} = \begin{pmatrix} -8 - x_0 \\ 4 - y_0 \end{pmatrix}$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -8 - x_0 \\ 4 - y_0 \end{pmatrix} + \begin{pmatrix} -8 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 + x_0 - 8 \\ -4 + y_0 + 4 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

i. Because two rotations of 180° will return any point to its original position

j. When n is odd $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -8 - x_0 \\ 4 - y_0 \end{pmatrix}$

k. (a, b) is invariant

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$

$$a = (\cos\theta)a - (\sin\theta)b + x \Rightarrow x = a(1 - \cos\theta) + b(\sin\theta)$$

$$b = (\sin\theta)a + (\cos\theta)b + y \Rightarrow y = b(1 - \cos\theta) - a(\sin\theta)$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} a(1 - \cos\theta) + b\sin\theta \\ b(1 - \cos\theta) - a(\sin\theta) \end{pmatrix}$$

TOPIC 4 SL WORKED SOLUTIONS

Topic 4 SL Paper 1, Group 1

1. **a.** €1.89 in Germany, €1.84 in Italy

b. One-tailed

c. **i.** H_0: The mean cost is the same in Germany and Italy.

H_1: The mean cost in Italy is less than the mean cost in Germany.

ii. p-value = 0.388

iii. p-value = 0.388 > 0.05, so insufficient evidence to reject H_0: The mean cost is the same.

2. There are 53 – 20 = 33 lockers in total.

a. The event "49 or higher" = {49, 50, 51, 52, 53} so the required probability is $\frac{5}{33}$

b. The event "26 or less" = {21, 22, 23, 24, 25, 26} so the required probability is $\frac{6}{33}$

c. The event "a multiple of 9" = {27, 36, 45} so the required probability is $\frac{3}{33}$

d. Apply the complementary event: the required probability is $1-\frac{3}{33}=\frac{30}{33}$

e. The event "a factor of 120" = {24, 30, 40} so the required probability is $\frac{3}{33}$

3. **a.** $4a+0.47+0.17+0.09+0.02+0.01=1$ hence $a=0.06$

b. $P(T=4|\,T\geq 3)=\dfrac{P((T=4)\cap(T\geq 3))}{P(T\geq 3)}=\dfrac{P(T=4)}{P(T\geq 3)}=\dfrac{0.09}{0.17+0.09+0.06+0.02+0.01}=\dfrac{9}{35}$

4. **a.** $a=12$, $b=7$

b. **i.** H_0: The choice of main course is independent of gender.

H_1: The choice of main course is not independent of gender.

ii. $\chi^2 = 4.0335\ldots \approx 4.03$

iii. 4.03 < 7.81 so the result is not significant at the 5% level so insufficient evidence to reject the null hypothesis that the choice of main course is independent of gender.

(degrees of freedom = 3)

5. **a.** 31

b.

Mid-value	Frequency
62.5	3
67.5	3
72.5	8
77.5	10
82.5	5
87.5	2

$\bar{x} = 75.2$ minutes, $\sigma = 6.58$ minutes

c. 105.2 minutes, 6.58 minutes

6. The sample space diagram shows the possible outcomes of the total T.

T	**2**	**3**	**5**	**8**
1	3	4	6	9
1	3	4	6	9
2	4	**5**	7	**10**
3	**5**	6	8	11
5	7	8	**10**	13
8	**10**	11	13	16

a. $P(T \text{ is prime}) = \frac{10}{24} = \frac{5}{12}$

b. $P(T \text{ is prime or a factor of } 10) = \frac{13}{24}$

c. $P(T \text{ is prime or a multiple of } 4) = \frac{16}{24}$

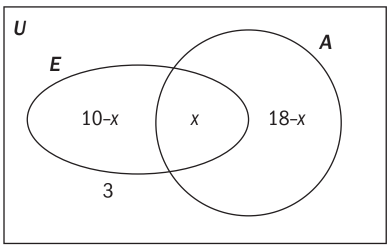
Topic 4 SL Paper 1, Group 2

7. **a.** $a = 4$

b. An outlier is any value greater than $1.5 \times 3.3 + 7.3 = 12.25$

So 13.2 should be shown as an outlier.

8. Represent the context with a Venn diagram



Hence write down the equation $10 - x + x + 18 - x + 3 = 30$, which has solution $x = 1$.

a. Using Venn diagram,

$P(\text{choose a student who studies both Economics and Art}) = \frac{1}{30}$

b. $P(\text{Art} \mid \text{Economics}) = \frac{P(\text{Art and Economics})}{P(\text{Economics})} = \frac{1}{10}$

c. $P(\text{Art}) \times P(\text{Economics}) = \frac{18}{30} \times \frac{10}{30} = \frac{1}{5}$

$\neq P(\text{choose a student who studies both Economics and Art}) = \frac{1}{30}$

Hence the events are not independent.

(Equivalently, $P(\text{Art}) = \frac{18}{30} \neq P(\text{Art} \mid \text{Economics})$ hence the events are not independent.)

9. **a.** A, B, C

b. B, D, C

10. $V \sim N(499.3, 3.7^2)$. Let v be the quantity required. Then $P(V < v) = 0.04$.
Use the inverse normal feature of your GDC to find $v = 493\,\text{ml}$.

11. **a.** **i.** $\frac{85}{200} \times 50 = 21.25 \approx 21$

ii. $50 - 21 = 29$

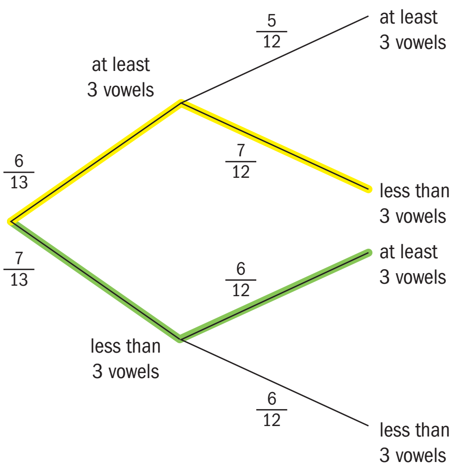
b. Randomly choose a student from grade 9 and then choose every $\frac{85}{21} \approx 4\text{th}$ student from that student using the list provided by the Principal.

c. The data is not symmetrical which indicates the population may not be normally distributed.

12. **a.** $P(\text{at least 3 vowels}) = \frac{4+1+1}{13} = \frac{6}{13}$

b. $25 \times \frac{6}{13} \approx 11.5$

c.



The event "exactly one of the two names chosen has at least 3 vowels" means either the first name chosen has at least three vowels and the second does not, or the first does not have at least three vowels and the second does. These combined events are highlighted in the diagram above.

Hence the required probability is $\frac{6}{13}\times\frac{7}{12}+\frac{7}{13}\times\frac{6}{12}=\frac{84}{156}=\frac{7}{13}$

Topic 4 SL Paper 1, Group 3

13. **a.** Let the score be u_n where $u_n=8+(n-1)2=6+2n$

b. $\bar{x}=\frac{S_n}{n}=\frac{\frac{n}{2}(16+2(n-1))}{n}=\frac{1}{2}(14+2n)=7+n>20$

$\Rightarrow n\geq 14$

$0.8 + 5p + 6q + 0.7 + 2.4 = 6.2$ so $5p + 6q = 2.3$

Solve the system to find $q = 0.3$ and $p = 0.1$.

14. **a.** $0.2 + p + q + 0.1 + 0.3 = 1$ so $p + q = 0.4$.

$0.8 + 5p + 6q + 0.7 + 2.4 = 6.2$ so $5p + 6q = 2.3$

Solve the system to find $q = 0.3$ and $p = 0.1$.

b. The ticket should cost USD 6.20 in order to make the expected gain zero.

c. The probability of winning at least USD 7 is 0.4. The number of times at least USD 7 is won in 10 games (X) is distributed binomially: $X\sim \mathrm{B}(10,0.4)$. The probability required is $P(X=4)=0.251$.

15. **a.** Ranking from lowest to highest

	A	B	C	D	E	F	G	H
Mass (m)	1	2	3	4	5	6	7	8
Height (h)	1	2	3	4	7	5.5	5.5	8

$r_s=0.922$

This is a strong correlation and so the height of the plants does generally increase with the amount of nutrients provided.

b. $h=0.826m+5.44$

c. The amount of growth in cm per gram of nutrient ($0.826\ \mathrm{cm\,g^{-1}}$)

d. $h=0.826\times 20+5.44=21.96$, so $c=21.96\approx 22.0$

16. **a.** $X\sim \mathrm{B}(20,0.4)$ and $Y\sim \mathrm{B}(100,0.08)$. In both cases, the assumption is that the trials are independent and the probability of scoring a point is fixed.

b. $\mathrm{E}(X)=20\times 0.4=8$ $\quad$ $\mathrm{E}(Y)=100\times 0.08=8$

c. $\mathrm{Var}(X)=20\times 0.4\times 0.6=4.8$ $\quad$ $\mathrm{Var}(Y)=100\times 0.08\times 0.92=7.36$

d. Since $\mathrm{E}(X)=\mathrm{E}(Y)$, the mean of the distributions are equal, meaning that on average the expected number of points scored for each distribution is equal.

Since $\mathrm{Var}(Y)$ is larger than $\mathrm{Var}(X)$, the number of points scored in Y will be spread more widely than when playing X.

17. **a.** $T \sim N(10, 2.5^2)$ $P(0 < T < 15) = 0.977$.

b. If Y represents the number of games at which Brian does not find his seat by the start of the game, then $Y \sim B(20, 0.022781748\ldots)$, using the unrounded answer from part (a) to avoid unnecessary rounding errors.

This is assuming that Brian attends each of the 20 games and that the probability of him finding his seat by the start of the game remains constant.

i. $P(Y \geq 2) = 0.0752$

ii. $P(Y = 2) = 0.0651$

18. **a.** H_0: X is distributed as $B(2, 0.5)$.

H_1: X is not distributed as $B(2, 0.5)$.

b. Assuming $B(2, 0.5)$

Number of times stopped	0	1	2
Frequency	$(0.5)^2 \times 60 = 15$	$2 \times 0.5 \times 0.5 \times 60 = 30$	$(0.5)^2 \times 60 = 15$

c. p-value = 0.0273 < 0.05. so sufficient evidence to reject H_0

d. $p > 0.5$ (lights are on red more than green), the incidents of red are not independent, the trials are not identical e.g. more traffic in the evening.

Topic 4 SL Paper 2

1. **a.** $a = 6.96103$, $b = -454.805$

$a = 6.96$, $b = -455$ (accept $6.96x - 455$)

b. $P = 6.96(270) - 455$

$= 1424.67$

$P = 1420$ (g)

c. 40 (hives)

d. **i.** 128 + 40

168 hives have a production less than k

$k = 1640$

ii. 200 − 168

32 (hives)

e. $X \sim B(n, p), \binom{n}{r} p^r (1 - p)^{n-r}$

$n = 40$ and $p = 0.75$ and $r = 30$

$P(X = 30) = 0.144364 \approx 0.144$

2. a. Let M denote the height of a randomly selected male.

Then $M \sim N\left(170, 11^2\right)$

$P(M > 175) = 0.325$

b. Let F denote the height of a randomly selected female.

Then $F \sim N\left(163, 10^2\right)$

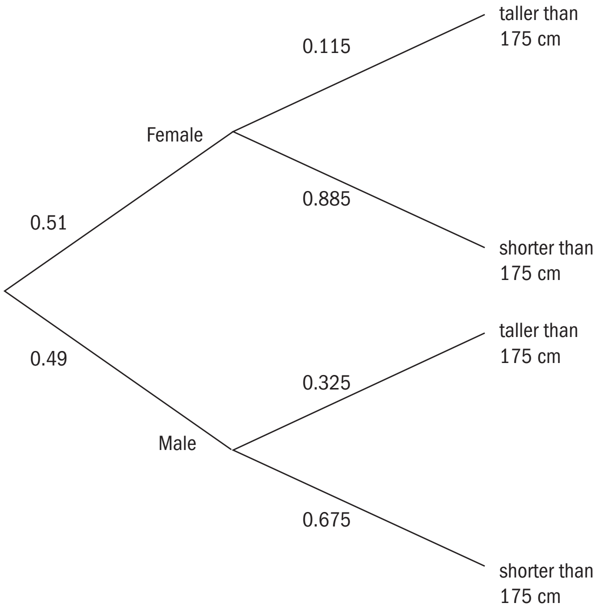
$P(F < t) = 0.15$. Use the inverse norm function of your GDC to find $t = 153\,\text{cm}$

c. The interquartile range is $\text{IQR} = Q_3 - Q_1$ where $P(M < Q_1) = 0.25$.

Hence $Q_1 = 162.58061275605$

Since the normal curve is symmetric, $\text{IQR} = 2 \times (170 - 162.58..) = 14.8\,\text{cm}$

d.



The tree diagram shows that the probability required is

$0.51 \times 0.115 + 0.49 \times 0.325 = 0.218$

e. Let B represent the event "the student is taller than 175 cm" and let A represent the event "the student is male".

The probability required is $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.49 \times 0.325}{0.218} = 0.731.$

TOPIC 4 HL WORKED SOLUTIONS

Topic 4 HL Paper 1, Group 1

1. **a.** A T distribution with 9 degrees of freedom or $T(9)$

b. $10.3 < \mu < 14.1$

2. **a.** $\mathrm{E}(A) = \mathrm{Var}(A) = 3.1$ and $\mathrm{E}(B) = \mathrm{Var}(B) = 2.7$

$\mathrm{E}(A-2B) = \mathrm{E}(A) - 2\mathrm{E}(B) = -2.3$

$\mathrm{Var}(A-2B) = \mathrm{Var}(A) + 4\mathrm{Var}(B) = 13.9$

b. Since $\mathrm{E}(A-2B)$ and $\mathrm{Var}(A-2B)$ are different, $A-2B$ cannot follow a Poisson distribution since the mean and the variance are always equal for every Poisson distribution.

3. **a.** The sample is large enough for the central limit theorem to apply

b. H_0: $\mu = 0$ $\qquad$ H_1: $\mu \neq 0$

$$s_{n-1} = \sqrt{\frac{40}{39}} \times 1.2 = 1.215...$$

p-value $= 0.00770... < 0.05$ so the null hypothesis that the mean is equal to 0 is rejected.

c. It is likely there is a systematic error that is causing Antoine to underestimate the results.

4. $\mathrm{E}.(X) = 4.9$, $\mathrm{E}(Y) = 7 \times 0.35 = 2.45$ and $\mathrm{E}(Z) = 61$.

$\mathrm{Var}(X) = 4.9$, $\mathrm{Var}(Y) = 7 \times 0.35 \times 0.65 = 1.5925$ and $\mathrm{Var}(Z) = 25$

a. $\mathrm{E}(2Y+7) = 2 \times 2.45 + 7 = 11.9$

b. $\mathrm{Var}(8-2X) = 4 \times \mathrm{Var}(X) = 19.6$

c. $\mathrm{Var}(X+2Y-Z) = \mathrm{Var}(X) + 4\ \mathrm{Var}(Y) + \mathrm{Var}(Z) = 36.27$

5. **a.** H_0: $p = 0.5$ $\qquad$ H_1: $p > 0.5$

b. Let X be the number Heads.

$P(X \geq a) < 0.05$, from the GDC, $a = 32$

Hence the critical region is $X \geq 32$

c. Accept H_0 if $X \leq 31$

$P(X \leq 31 \mid p = 0.6) = 0.664$

6. The unbiased estimate of the population mean is the mean of the sample which is 248.3.

The unbiased estimate of the population variance is found from the GDC as $6.566\,751\,268\,490\,5^2 = 43.1$, or equivalently $6.229\,767\,250\,869^2 \times \dfrac{10}{9} = 43.1$.

Topic 4 HL Paper 1, Group 2

7. Using the χ^2 goodness of fit test

sample mean = 1.26315…

H_0: Distribution is Poisson. H_1: Distribution is not Poisson.

Expected values for Po(1.26)

Number of goals	0	1	2	3	4	≥5
Frequency	10.7	13.6	8.57	3.61	1.14	0.38

Need to combine the final 3 columns for the observed and expected tables.

Number of goals	0	1	2	≥3
Observed	9	14	12	3
Expected	10.7	13.6	8.57	5.13

Degrees of freedom $= 4 - 1 - 1 = 2$

p-value $\approx 0.28 > 0.1$, not significant even at the 10% level so insufficient evidence to reject H_0.

8. **a.** Let V denote the number of admissions to the emergency room during Vicky's shift.

Assuming the average rate is the same, $V \sim \text{Po}(4 \times 4.7)$.

Hence $P(V \geq 20) = 0.421$

b. Let S denote the number of Saturday evening shifts on which there are at least 20 admissions to the emergency room. Then assuming that 0.421 is constant for all 5 shifts, and that the number of admissions on each evening is independent, then $S \sim \text{B}(5, 0.421)$. $P(S = 2) = 0.344$ is the maximum of the probabilities, so the most likely number of evenings is 2.

9. **a.** Po(8.1), this is assuming that the number of problems in office A is independent of the number of problems in office B.

b. H_0: $\mu = 2 \times 8.1 = 16.2$ H_1: $\mu < 16.2$

$P(X \leq 12 \mid \mu = 16.2) \approx 0.180 > 0.05$

Hence insufficient evidence at the 5% level to reject H_0 that the mean rate has stayed the same.

10. **a.** Let F denote the number of flaws in a silk sheet of area 10 square metres.

Then $F \sim \text{Po}(1.8)$. $P(F \geq 1) = 0.8347$

b. Let R denote the profit from one silk sheet of area 10 square metres.

Then $\text{E}(R) = 350 \times P(F = 0) + 150 \times P(F = 1) - 100 \times P(F \geq 2) = €48.77$

Hence the expected profit from 70 sheets is €3400 to the nearest 100 Euros.

11. a. Quota

b. $\frac{20}{50} = 0.40$

c. $\text{H}_0\text{: } p = 0.2 \qquad \text{H}_1\text{: } p > 0.2$

Let X be the number of people who respond to the survey and under H_0 assume $X \sim \text{B}(50, 0.2)$

$P(X \geq 20) = 0.000\,932 < 0.01$

Hence, sufficient evidence to reject H_0 and to accept the alternative that the proportion of people who would visit the coffee shop is greater than 0.2

d. The question is too vague. It does not ask how often they might visit the coffee shop.

12. a. Let $\bar{A}$ denote the mean height of the 70 corn plants fertilized with type A.

Then $\bar{A} \sim \text{N}\left(253, \frac{10^2}{70}\right)$. Hence $P\left(\bar{A} > 255\right) = 0.0471$.

b. Let $\bar{B}$ denote the mean height of the 80 corn plants fertilized with type B.

Then $\bar{B} \sim \text{N}\left(250, \frac{12^2}{80}\right)$. The required probability is $P\left(\bar{A} > \bar{B}\right) = P\left(\bar{A} - \bar{B} > 0\right)$.

Let $D = \bar{A} - \bar{B}$. Then $D \sim \text{N}\left(253 - 250, \frac{10^2}{70} + \frac{12^2}{80}\right)$. Hence $P(D > 0) = 0.953$.

Topic 4 HL Paper 1, Group 3

13. a. $P\left(\bar{X} < a\right) = 0.05$

$\bar{X} \sim \text{N}\left(1000, \frac{50}{\sqrt{15}}\right)$

$a \approx 979$

b. H_0 is accepted if $\bar{X} > 979$ g

$P\left(\bar{X} > 979 \mid \mu = 980\right) = 0.531$

c. The number of times the machine fails is B(3, 0.469…)

$\left(1 - (1 - 0.469...)^3\right) \approx 0.850$

14. a.

$$\mathbf{T} = \begin{array}{c c} & \begin{array}{ccc} & \text{Current phone} & \\ \text{Pi} & \text{Mu} & \text{Fi} \end{array} \\ \text{Future phone} \begin{array}{c} \text{Pi} \\ \text{Mu} \\ \text{Fi} \end{array} & \begin{pmatrix} 0.76 & 0.11 & 0.2 \\ 0.10 & 0.7 & 0.05 \\ 0.14 & 0.19 & 0.75 \end{pmatrix} \end{array}$$

b. $\begin{pmatrix} 0.76 & 0.11 & 0.2 \\ 0.10 & 0.7 & 0.05 \\ 0.14 & 0.19 & 0.75 \end{pmatrix}^3 \begin{pmatrix} 0.32 \\ 0.4 \\ 0.28 \end{pmatrix} = \begin{pmatrix} 0.378 \\ 0.25 \\ 0.372 \end{pmatrix}$.

Hence Fi's claim is not correct because Pi are predicted to have a slightly larger share of the market, assuming that the current transition probabilities remain applicable for the three years.

15. a. i. The curve has a single maximum.

ii. The curve has a maximum and is not symmetric about the maximum.

b. For $y = -0.4x^2 + 2.9x$

$$SS_{res} = (2.5-2)^2 + (5.1-4.5)^2 + (4.5-4.3)^2 + (3.0-1.7)^2$$

$$= 0.5^2 + 0.6^2 + 0.2^2 + 1.3^2 \approx 2.34$$

For $y = -0.155x^3 + 0.944x^2$

$$SS_{res} = (0.789-2)^2 + (4.311-4.5)^2 + (4.225-4.3)^2 + (0.504-1.7)^2$$

$$= 1.21^2 + 0.189^2 + 0.075^2 + 1.196^2 \approx 2.94$$

Hence $y = -0.4x^2 + 2.9x$ is the better fit.

c. $y = -0.0992x^3 + 0.555x^2 + 0.319x + 1.23$

d. $R^2 = 1$, the curve is a perfect fit to the points.

e. When $t = 0$ the height is 0 for both the given curves but is equal to 1.23 for the regression curve.

16. a. T denote the total mass of the 5 randomly chosen strawberries.

Then $T = T_1 + T_2 + T_3 + T_4 + T_5$.

Consequently $T \sim \mathrm{N}(60, 5 \times 2.7^2)$. Hence $P(T \geq 70) = 0.0488$

b. Let S denote the mass of the randomly chosen strawberry and R denote the mass of the randomly chosen raspberry.

Then the probability required is $P(S > 4R) = P(S - 4R > 0)$.

Consider the random variable $Y = S - 4R$. Assuming S and R are independent,

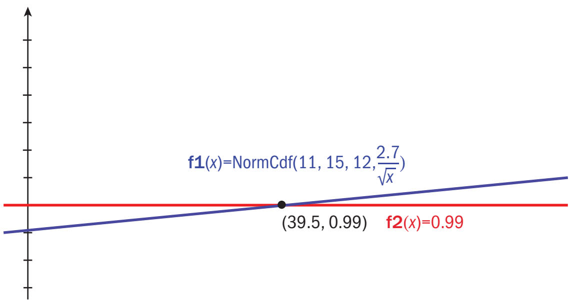
$$Y \sim \mathrm{N}(12 - 4 \times 4, 2.7^2 + 16 \times 0.5^2), \text{ giving } Y \sim \mathrm{N}(-4, 11.29).$$

Therefore the probability required is 0.3616.

c. Let $\bar{X}$ denote the mean weight of the n strawberries. Then $\bar{X} \sim \mathrm{N}\left(12, \dfrac{2.7^2}{n}\right)$.

If the probability the container is rejected is 0.01 then $P(11 < \bar{X} < 15) = 0.99$.

Applying technology it can be seen that $n = 39$ or that $n = 40$ both give the required probability to two decimal places.



17. **a.** Total number of posters accepted = 124, proportion = $\frac{124}{250} = 0.496$

(or $\bar{x} = 5p = 2.48 \Rightarrow p = 0.496$)

b.

Number of posters accepted	0	1	2	3	4	5
Expected frequency	1.626	8.001	15.748	15.498	7.626	1.501

Combining columns:

Number of posters accepted	0&1	2	3	4&5
Observed frequency	15	10	7	18
Expected frequency	9.627	15.748	15.498	9.127

H_0: The results follow a binomial distribution.

H_1: The results do not follow a binomial distribution.

Degrees of freedom = 4 −1 −1 = 2

p-value = 0.000 102 < 0.05 so the null hypothesis is rejected.

The results do not follow a binomial distribution and hence this is likely to mean that the selection of posters is not independent of the sample they are in.

18. **a.** Assuming that in the following season the sunflowers grow according to the same distribution, then the heights of the sunflowers H follow a normal distribution $H \sim N(189.5, 15.3^2)$.

Hence the expected number of sunflowers growing higher than 195 cm is

$310 \times P(H > 195) = 111.$

b. The heights Izzy measured were 3 cm too large.

Consider the random variable $C = H - 3$.

This would transform the data to the correct values for the heights of the sunflowers.

$E(H-3) = 189.5 - 3 = 186.5$

$\mathrm{Var}(H-3) = \mathrm{Var}(H) = 15.3^2 = 234$

c. The true answer to (a) is therefore 90.

Topic 4 HL Paper 2

1. **a.** 0.916, a strong positive correlation

b. H_0: $\rho = 0$, H_1: $\rho \neq 0$

p-value = 0.0290 < 0.05 hence sufficient evidence at the 5% significance level to reject H_0 and to accept that there is sufficient evidence of correlation between the results of the two surveys.

c. Test-retest

d. This is a paired-sample t-test.

Employee	A	B	C	D	E
First survey	7.2	4.1	6.1	5.4	3.9
Second survey	7.3	5.2	6.3	6.7	4.2
Difference	0.1	1.1	0.2	1.3	0.3

Let μ_D be the mean difference.

H_0: $\mu_D = 0$, H_1: $\mu_D > 0$

p-value = 0.0368 < 0.05, hence sufficient evidence at the 5% level to accept H_1 that the cafeteria food has improved.

2. **a.** Let H denote the weight of a randomly chosen item of hand luggage.

Then $H \sim N(9.4, 2.8^2)$. Hence $P(9 < H < 12) = 0.380$

b. Assuming that the weights of the 8 items of hand luggage are independent of each other, let T denote the total weight of 8 randomly chosen items of hand luggage.

Then $T \sim N(8 \times 9.4, 8 \times 2.8^2)$.

Hence $P(T > 100) = 0.000\ 870$.

c. Let L denote the weight of a randomly chosen large case.

Then $L \sim N(25, 7^2)$.

The event is $P(L > 3H) = P(L - 3H > 0)$.

The random variable $(L - 3H)$ has distribution

$(L - 3H) \sim N(25 - 3 \times 9.4,\ \ 2.8^2 + 9 \times 7^2)$.

Hence $P(L - 3H > 0) = 0.440$.

d. Let S denote the total weight of the randomly chosen sample of three large cases and two items of hand luggage.

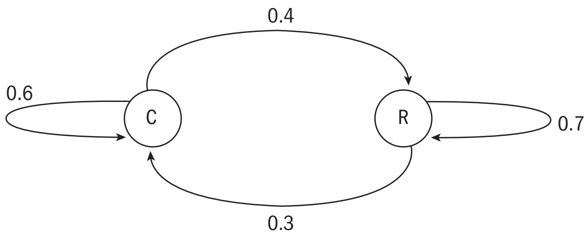
Then $S \sim N(2 \times 9.4 + 3 \times 25, 2 \times 2.8^2 + 3 \times 7^2)$.

The probability required is $P(S > 105) = 0.190$.

Topic 4 HL Paper 3

1. **a.** This context can be modelled with a Markov chain because the probabilities of transitioning from each current state to each future state depend what the current state is.

b.



c.

$$\mathbf{G} = \begin{array}{cc} & \begin{array}{cc} & \text{Current state} \\ & \begin{array}{cc} \text{C} & \text{R} \end{array} \end{array} \\ \begin{array}{c} \text{Future} \\ \text{state} \end{array} \begin{array}{c} \text{C} \\ \text{R} \end{array} & \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix} \end{array}$$

d. $\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}\begin{pmatrix} 17\,530 \\ 8956 \end{pmatrix} = \begin{pmatrix} 13\,204.8 \\ 13\,281.2 \end{pmatrix}$, so after one year, ClearGym will have 13 200 customers and ResultsNow will have 13 300 customers, to the nearest 100 customers.

e. $$\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0.6-\lambda & 0.3 \\ 0.4 & 0.7-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow (0.6-\lambda)(0.7-\lambda)-0.12=0$$

$$\Rightarrow 10\lambda^2-13\lambda+3=0 \Rightarrow \lambda=0.3 \text{ or } \lambda=1$$

$$\lambda=1 \Rightarrow \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 0.4x-0.3y=0 \Rightarrow 3y=4x$$

Hence the eigenvector associated with $\lambda=1$ is $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

$$\lambda=0.3 \Rightarrow \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.3\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow x+y=0 \Rightarrow y=-x$$

Hence the eigenvector associated with $\lambda=0.3$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

f. Hence $\mathbf{G} = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix}\begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{-3}{7} \end{pmatrix}$

g. $$\mathbf{G}^n = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix}^n\begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{-3}{7} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1^n & 0 \\ 0 & 0.3^n \end{pmatrix}\begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{-3}{7} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0.3^n \\ 4 & -0.3^n \end{pmatrix}\begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{-3}{7} \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 3 & 0.3^n \\ 4 & -0.3^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -3 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 3+4(0.3^n) & 3-3(0.3^n) \\ 4-4(0.3^n) & 4+3(0.3^n) \end{pmatrix}$$

h. The number of customers each gym has after one year is

$$\frac{1}{7}\begin{pmatrix} 3+4(0.3^n) & 3-3(0.3^n) \\ 4-4(0.3^n) & 4+3(0.3^n) \end{pmatrix}\begin{pmatrix} 17\,530 \\ 8956 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 4.2 & 2.1 \\ 2.8 & 4.9 \end{pmatrix}\begin{pmatrix} 17\,530 \\ 8956 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}\begin{pmatrix} 17\,530 \\ 8956 \end{pmatrix}$$

which is exactly the same expression as in (d).

i. In the long term, $\mathbf{G}^n = \frac{1}{7}\begin{pmatrix} 3+4(0.3^n) & 3-3(0.3^n) \\ 4-4(0.3^n) & 4+3(0.3^n) \end{pmatrix} \to \frac{1}{7}\begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & \frac{3}{7} \\ \frac{4}{7} & \frac{4}{7} \end{pmatrix}$.

Since $\begin{pmatrix} \frac{3}{7} & \frac{3}{7} \\ \frac{4}{7} & \frac{4}{7} \end{pmatrix}\begin{pmatrix} 17\,530 \\ 8956 \end{pmatrix} = \begin{pmatrix} 11351 \\ 15135 \end{pmatrix}$ to the nearest customer, this shows

that the management of ClearGym is justified in having this concern, because the prediction is that they begin with 17 530 customers compared to ResultsNow's 8956, but the long term prediction is that they will have 11 351 customers which is a reduction on the intial number of customers.

Because of this, ResultsNow's number of customers increases from 8956 to 15 135.

TOPIC 5 SL WORKED SOLUTIONS

Topic 5 SL Paper 1, Group 1

1. **a.** $\frac{dy}{dx} = 8x^3 - 7$

b. $8 \times (-1)^3 - 7 = -15$

c. Using $y - y_1 = m(x - x_1)$

$y - 9 = -15(x - (-1)) \Rightarrow y - 9 = -15x - 15$

$15x + y + 6 = 0$

2. **a.** $\int_1^2 \frac{1}{x}\,dx = 0.6931... \approx 0.693$ (3 sf)

b. $h = 0.25$

x	1	1.25	1.5	1.75	2.0
y	1	0.8	0.666…	0.5714…	0.5

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{0.25}{2}\big((1+0.5) + 2(0.8 + 0.666... + 0.5714...)\big) = 0.6970... \approx 0.697$$

c. Percentage error $= \frac{0.6970... - 0.6931...}{0.6931...} \times 100\% \approx 0.559\%$

3. $y = \int 3x^2 - 8x + 1\,dx = x^3 - 4x^2 + x + c$

Passes through $(2,7)$ so $7 = 8 - 4 \times 4 + 2 + c$

$\Rightarrow c = 13$

$y = x^3 - 4x^2 + x + 13$

4. **a.** $\frac{dy}{dx} = 4x^3 - 32$

b. $4x^3 - 32 = 0$

c. $x^3 = 8 \Rightarrow x = 2$

$y = 2^4 - 32 \times 2 + 8 = -40$

$P = (2, -40)$

5. Horizontal distance between points, h, is equal to 1

Area of top half of fish $\approx \frac{1}{2}\big((0 + 1.8) + 2(1.4 + 1.8 + 2 + 1.8 + 1.5 + 0.5)\big)$

$= 9.9\ \text{cm}^2$

Whole area $\approx 2 \times 9.9 = 19.8\ \text{cm}^2$

6. a. To find the width and height, enter the equation into your calculator and find the x-intercept and maximum point.

Width = 6 m, height = 5.4 m

b. Area of cross-section $\int_0^6 3.6x - 0.6x^2\,\mathrm{d}x = 21.6\text{ m}^2$

Volume = $21.6 \times 50 = 1080\text{ m}^3$

Topic 5 SL Paper 1, Group 2

7. a. $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x^3 - 2x^2 - 6$

b. $2x^3 - 2x^2 - 6 = 30$

$x = 3$

$y = \frac{1}{2}(3)^4 - \frac{2}{3}(3)^3 - 6(3) - 3 = 1.5$

$P = (3, 1.5)$

8. a. $\frac{\mathrm{d}C_A}{\mathrm{d}x} = -6x^{-2}$

$C_A = -\int 6x^{-2}\,\mathrm{d}x = 6x^{-1} + c$

$4.5 = \frac{6}{12} + c \Rightarrow c = 4$

$C_A = 4 + \frac{6}{x}$

When $x = 15$, $C_A = 4 + \frac{6}{15} = 4.4$

b. $C = x\left(4 + \frac{6}{x}\right) = 4x + 6$

9. If the function is continuous, the value of both parts must be equal when $x = 4$

$f(4) = 0.8 \Rightarrow 4a + b = 0.8$

If the curves have the same gradient at $x = 4$ their derivatives must be equal.

$$f'(x) = \begin{cases} a & 0 \le x < 4 \\ 0.4x - 2.4 & 4 \le x \le 8 \end{cases}$$

$f'(4) = 0.4 \times 4 - 2.4 = -0.8 \Rightarrow a = -0.8$

$b = 0.8 + 4 \times 0.8 = 4$

10. a. $240 = 0.6q^2 \Rightarrow q^2 = 400$

$\Rightarrow q = 20$ units

b. $\int_0^{20} 0.6q^2\,\mathrm{d}q = 1600$

c. Area of S = $240 \times 20 - 1600 = 3200$

11. a. $f'(x) = 6x^2 - 18x + 12$

b. Either by factorising or obtaining from a graph on the GDC

$x = 1, 2$

c. Either solving $6x^2 - 18x + 12 < 0$ on the GDC or by graphing $f(x) = 2x^3 - 9x^2 + 12x + 1$ and observing where it is decreasing

$1 < x < 2$

d. i. Because when $x = 1$ the point is only a local maximum and the actual maximum occurs at $x = 3$

ii. $f(3) = 10$

12. a. $n = -\frac{1}{50}(20)^3 + (20)^2 + 50 = 290$

b. $\frac{dn}{dT} = -\frac{3}{50}T^2 + 2T$

c. Plot $\frac{dn}{dT} = -\frac{3}{50}T^2 + 2T$ on your calculator and find its maximum value, which is 16.7 °C.

d. It is where the graph is steepest.

Topic 5 SL Paper 1, Group 3

13. a. $f(x) = 2x + 3x^{-1} - 2$

$f'(x) = 2 - 3x^{-2}$

b. $2 - \frac{3}{x^2} = -4$

$\frac{3}{x^2} = 6 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \sqrt{\frac{1}{2}}$ (as $x > 0$)

14. a. i. Intercepts of $f(x) = 4x - x^2$ with x-axis are 0 and 4

$AD = (4 - x) - x = 4 - 2x$

ii. Area $= R = (4 - 2x)(4x - x^2)$ or $16x - 12x^2 + 2x^3$

b. Find the maximum value of $R = 16x - 12x^2 + 2x^3$ on your calculator, which is 6.16.

c. i. $\frac{dR}{dx} = 16 - 24x + 6x^2$

ii. $16 - 24 + 6 = -2$

15. a. $a(2)^3 + b(2)^2 + c(2) = 2$

$\Rightarrow 8a + 4b + 2c = 2$

b. $a(1)^3+b(1)^2+c(1)=-5$

$a+b+c=-5$

$f'(x)=3ax^2+2bx+c$

$3a+2b+c=0$

c. Solving on the GDC:

$a=1,\ b=3,\ c=-9$

16. **a.** $f'(x)=\frac{2}{3}x$

When $x=3$ $f'(3)=2$, gradient of normal $=-\frac{1}{2}$

Passes through $(3,3)$ so $y-3=-\frac{1}{2}(x-3) \Rightarrow y=-\frac{1}{2}x+\frac{9}{2}$

$x+2y-9=0$

b. $y=0 \Rightarrow x=9$

c. **i.** $A_R=\int_0^3 \frac{1}{3}x^2\mathrm{d}x+\int_3^9 -\frac{1}{2}x+\frac{9}{2}\mathrm{d}x$

ii. GDC $\Rightarrow$ Area $R=12$

17. **a.** **i.** $a \approx 16.7,\ b \approx 37.0$

ii. Hence, the maximum height of the roller coaster is 37 m

b. $h'(x)=\frac{3x^2}{500}-\frac{2x}{5}+5$

c. The steepest gradient is the minimum of $h'(x)$, from the GDC this is $-1.666\ldots$, (when $x \approx 33.3$ m)

d. Let the angle with the horizontal be α

$\tan\alpha = 1.666...$

$\Rightarrow \alpha \approx 59.0°$

18. **a.** $f'(x)=3x^2-5x-2=0 \Rightarrow x=-\frac{1}{3}, 2$

x-coordinate of minimum is 2

b. $f(x)=\int 3x^2-5x-2\mathrm{d}x = x^3-\frac{5}{2}x^2-2x+c$

Passes through $(0,1.5)$, so $c=1.5$

$f(x)=x^3-\frac{5}{2}x^2-2x+1.5$

c. One value of k is $f(2)=-4.5$

A second value of k is local maximum = $1.8518\ldots \approx 1.85$

Topic 5 SL Paper 2

1. a. $V = 50 = \pi r^2 h \Rightarrow h = \dfrac{50}{\pi r^2}$

b. $C = 8 \times 2\pi r^2 + 6 \times 2\pi r h = 16\pi r^2 + 12\pi r \times \dfrac{50}{\pi r^2}$

$= 16\pi r^2 + \dfrac{600}{r}$

c. $C = 16\pi r^2 + 600 r^{-1}$

$\dfrac{dC}{dr} = 32\pi r - \dfrac{600}{r^2}$

d. $32\pi r - \dfrac{600}{r^2} = 0$

e. $r = 1.8139... \approx 1.81$

f. $h = \dfrac{50}{\pi \times (1.81...)^2} \approx 4.84$ cm

2. a. $x \times \dfrac{600}{x^2} = \dfrac{600}{x}$

b. $0.75n$

c. $P = \dfrac{600}{x} - 0.75n$

$= \dfrac{600}{x} - 0.75 \times \dfrac{600}{x^2} = \dfrac{600}{x} - \dfrac{450}{x^2}$

d. $\dfrac{dP}{dx} = -\dfrac{600}{x^2} + \dfrac{900}{x^3}$

e. $-\dfrac{600}{x^2} + \dfrac{900}{x^3} = 0 \Rightarrow \dfrac{9}{x^3} = \dfrac{6}{x^2}$

$\Rightarrow x = 1.5$, so sell for 1.50 euros

$n = \dfrac{600}{1.5^2} \approx 267$ donuts

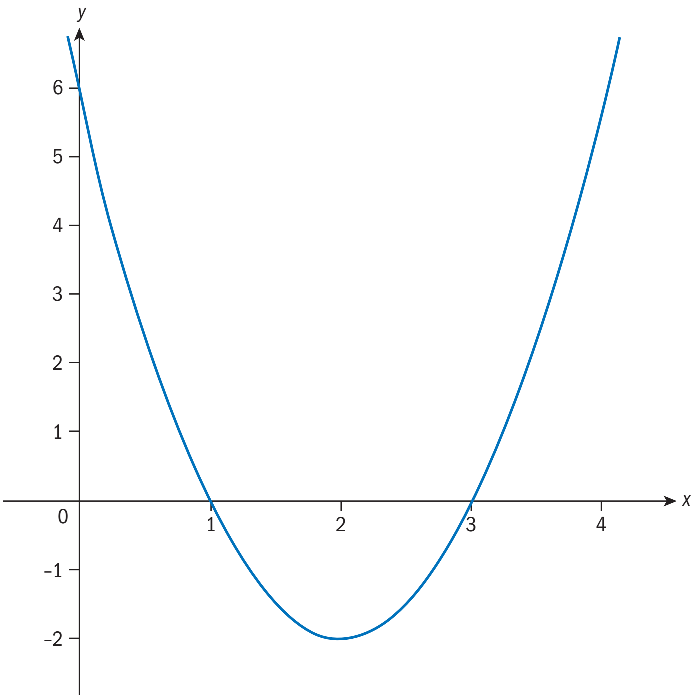
TOPIC 5 HL WORKED SOLUTIONS

Topic 5 HL Paper 1, Group 1

1. a. $f'(x) = \dfrac{e^x \times 2x - x^2 e^x}{(e^x)^2} = \dfrac{xe^x(2-x)}{(e^x)^2} = \dfrac{x(2-x)}{e^x}$

b. $f'(x) > 0$ when $0 < x < 2$

2. a.



b. $\int_0^4 2(x-1)(x-3)\,dx = 2.67$

c. $\int_0^4 |2(x-1)(x-3)|\,dx = 8$

3. a. $\dfrac{dM}{dt} = -kM \Rightarrow \int \dfrac{1}{M}\,dM = \int -k\,dt$

$\ln M = -kt + c \Rightarrow M = e^{-kt+c} = Ae^{-kt}$

When $t = 0$, $M = 50\text{g} \Rightarrow 50 = Ae^0 = A$

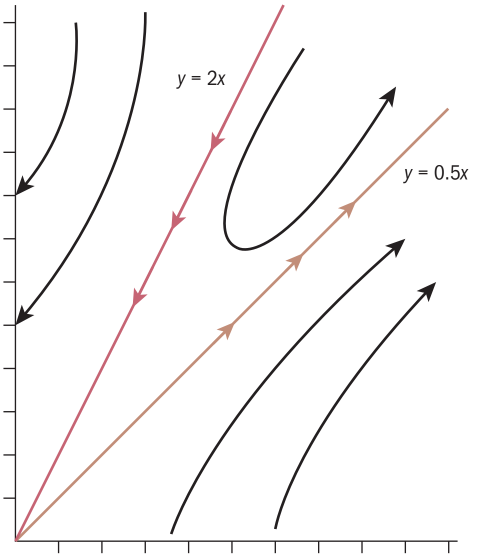
$M = 50e^{-kt}$

b. $25 = 50e^{-400k}$

Either solve directly on a GDC to get $k \approx 0.00173$

Or $0.5 = e^{-400k} \Rightarrow k = -\dfrac{1}{400}\ln 0.5 \approx 0.00173$

4. a.



b. The number of weevils needs to be in the region defined above or equal to the line $y = 2x$ so they need at least $2 \times 150 = 300$ weevils.

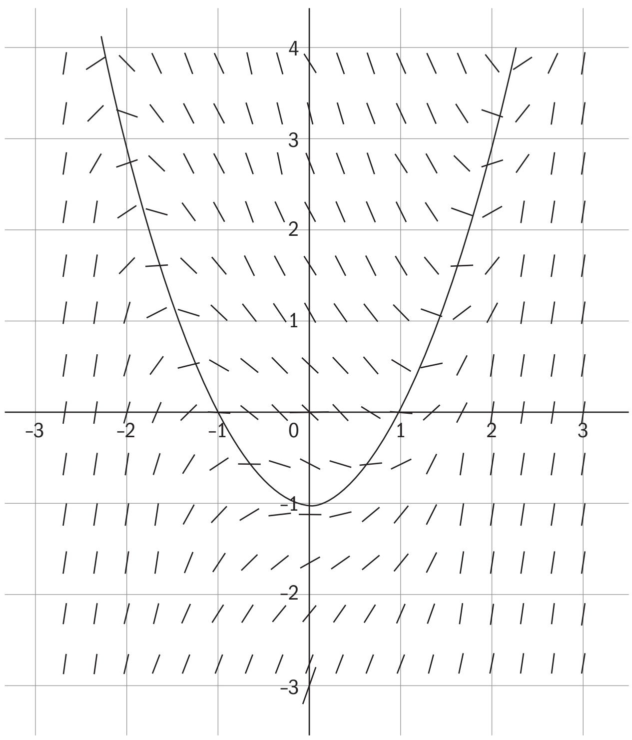
5. a. $\frac{dy}{dx} = 2 \times \frac{1}{2x-3} = \frac{2}{2x-3}$

b. When $x = 2$, $\frac{dy}{dx} = \frac{2}{4-3} = 2$, so the gradient of the normal is $-\frac{1}{2}$.

$y = -\frac{1}{2}x + c$, the normal passes through $(2,0)$ and hence $0 = -\frac{1}{2} \times 2 + c \Rightarrow c = 1$

6. a. i. $\frac{dy}{dx} = x^2 - y - 1 = 0 \Rightarrow y = x^2 - 1$

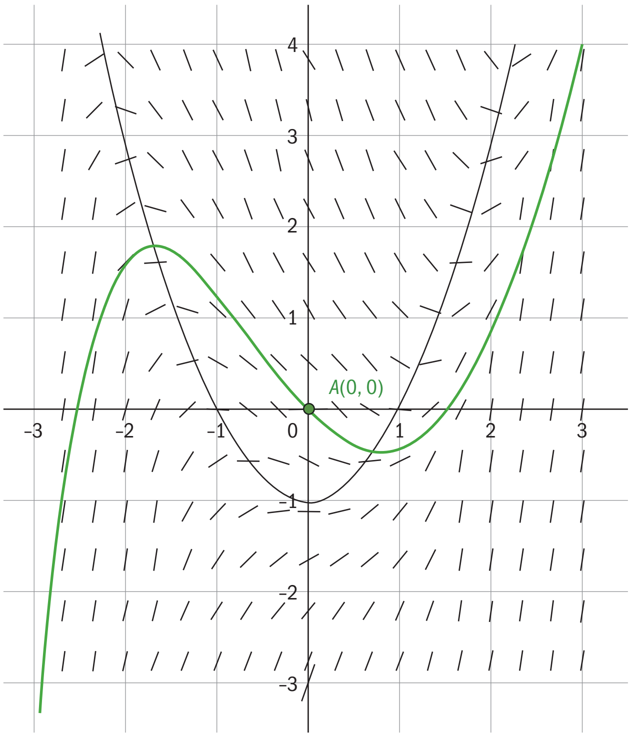
ii



b. $x_{n+1} = x_n + 0.2$, $y_{n+1} = y_n + 0.2\left(x_n^2 - y_n - 1\right)$

When $x = 2$, $y \approx 0.71410065\ldots \approx 0.714$

c. When sketching, ensure that the maximum and minimum points lie on the curve $y = x^2 - 1$. The intermediate values calculated using Euler's method would be helpful in the construction.



Topic 5 HL Paper 1, Group 2

7. All answers are obtained directly from the GDC, using the equation, numerical derivative and integration functions.

a. 2.95 seconds

b. $v'(2) \approx 0.659$

c. $\int_0^5 \left|1.4^t - 2.7\right| \mathrm{d}t \approx 5.35$

8. **a.** $V = \pi\int_0^1 \left(-0.2(y-0.5)^2 + 0.5\right)^2 \mathrm{d}y = 0.7346\ldots \approx 0.735\ \mathrm{m}^3$

b. $V = \int 0.6\mathrm{e}^{-t}\,\mathrm{d}t = -0.6\mathrm{e}^{-t} + c$

At $t = 0$, $V = 0$ and hence , hence $c = 0.6$ $V = 0.6 - 0.6\mathrm{e}^{-t}$

c. Limit for the volume is 0.6, hence maximum percentage filled is

$$\frac{0.6}{0.7346\ldots} \times 100\% \approx 81.7\%$$

9. **a.** $\dfrac{\mathrm{d}N}{\mathrm{d}t} = 1 \times \mathrm{e}^{-pt} - t \times p\mathrm{e}^{-pt} = \mathrm{e}^{-pt}(1 - pt)$

b. $\mathrm{e}^{-pt}(1 - pt) = 0 \Rightarrow 1 - pt = 0$ (as e^{-pt} is never 0)

$t = \dfrac{1}{p}$ and $N = \dfrac{1}{p}\mathrm{e}^{-p\times\frac{1}{p}} = \dfrac{\mathrm{e}^{-1}}{p} = \dfrac{1}{p\mathrm{e}}$

c. $\frac{d^2N}{dt^2} = e^{-pt} \times -p + \left(-pe^{-pt}(1-pt)\right) = e^{-pt}\left(-p-p+p^2t\right)$

$= e^{-pt}\left(p^2t - 2p\right)$

d. When $t = \frac{1}{p}$, $\frac{d^2N}{dt^2} = e^{-p\times\frac{1}{p}}\left(p^2 \times \frac{1}{p} - 2p\right) = -pe^{-1} < 0$ for all $p > 0$, hence the curve is concave down so the point is a maximum.

10. a. $\frac{dy}{dx} = 2 \times \frac{1}{\cos^2(2x-4)} = \frac{2}{\cos^2(2x-4)}$

b. $\frac{dy}{dx} = 2\left(\cos(2x-4)\right)^{-2}$

$\frac{d^2y}{dx^2} = 2 \times (-2)\left(\cos(2x-4)\right)^{-3} \times \left(-2\sin(2x-4)\right)$

$= \frac{8\sin(2x-4)}{\left(\cos(2x-4)\right)^3}$

c. For a point of inflection, $\frac{8\sin(2x-4)}{\left(\cos(2x-4)\right)^3} = 0 \Rightarrow \sin(2x-4) = 0$

$\Rightarrow x = 0.4292..., 2,...$ Hence $a \approx 0.429$

11. a. Area = $\int_0^a x^2\,dx = \left[\frac{1}{3}x^3\right]_0^a = \frac{1}{3}a^3$

b. Either: consider the rectangle formed by the lines $y = a^2$ and $x = a$.

Area = area of rectangle – area of R

$a^3 - \frac{1}{3}a^3 = \frac{2}{3}a^3$

Or: $x = y^{\frac{1}{2}}$ so area = $\int_0^{a^2} y^{\frac{1}{2}}\,dy = \left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^{a^2} = \frac{2}{3}\left(a^2\right)^{\frac{3}{2}} = \frac{2}{3}a^3$

12. a. $v = 4te^{t^2} \Rightarrow a = 4e^{t^2} + 4t \times 2te^{t^2}$

$= e^{t^2}\left(4 + 8t^2\right)$

b. $s = \int 4te^{t^2}\,dt$ Let $u = t^2 \Rightarrow \frac{du}{dt} = 2t$

$s = \int 4te^{u}\frac{du}{2t} = \int 2e^u\,du = 2e^u + c = 2e^{t^2} + c$

Topic 5 HL Paper 1, Group 3

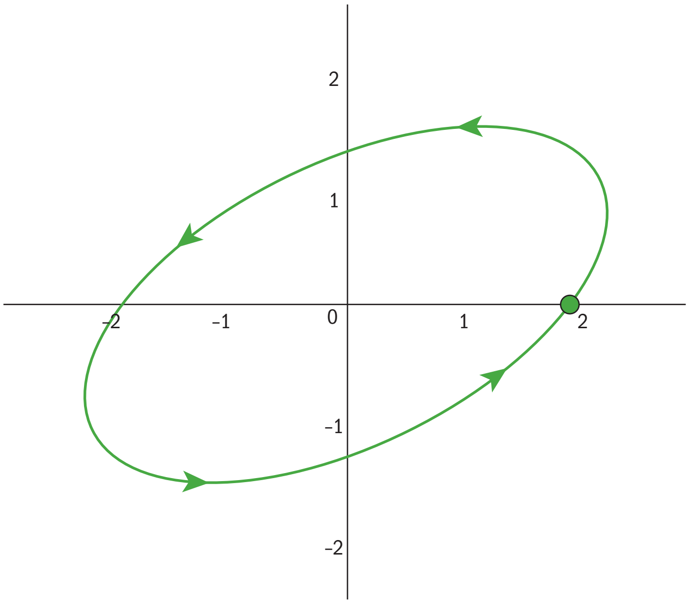
13. a. i. $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{4x-3y}{3x-9y}$, when $y = 0$, $\frac{dy}{dx} = \frac{4x}{3x} = \frac{4}{3}$

ii. When $x = 0$ $\frac{dy}{dx} = \frac{-3y}{-9y} = \frac{1}{3}$

b. $\det\begin{pmatrix} 3-\lambda & -9 \\ 4 & -3-\lambda \end{pmatrix} = 0$, $(3-\lambda)(-3-\lambda) + 36 = 0$

$\Rightarrow \lambda^2 - 9 + 36 = 0 \Rightarrow \lambda^2 = -27 \Rightarrow \lambda = \pm 3\sqrt{3}i$

c.



Note: Your sketch should be centered on (0, 0), passing through (2, 0) with a gradient of approximately $\frac{4}{3}$ as it crosses the x-axis and $\frac{3}{4}$ on the y-axis.

To find the direction note that $\dot{y} = 8$ at (2,0) so the trajectory is 'upwards' at this point.

14. a. $f(t) = 350\left(1 + 6e^{-t}\right)^{-1}$

$$f'(t) = -350\left(1 + 6e^{-t}\right)^{-2} \times -6e^{-t} = \frac{2100e^{-t}}{\left(1 + 6e^{-t}\right)^2}$$

b. i. 350 **ii.** 50

c. Maximum value of $f'(t)$ can be found from the GDC

$t = 1.7917... \approx 1.79$, rate of increase is 87.5 goats per year

d. When $t = 1.7917...$, population = 175

[Note: in the logistic model the greatest rate of increase always occurs when the population is half of the carrying capacity.]

15. a. $v = \begin{cases} t^2 + c_1 & 0 \le t < 3 \\ 6t + c_2 & t \ge 3 \end{cases}$

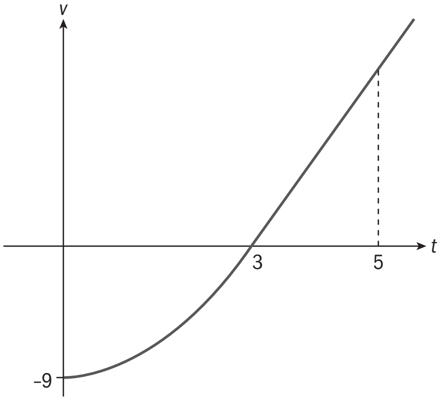
when $t = 0, v = -9 \Rightarrow c_1 = -9$

velocity is continuous at $t = 3$ so both branches are equal at $t = 3$:

When $t = 3, v = 3^2 - 9 = 0 \Rightarrow 0 = 6 \times 3 + c_2 \Rightarrow c_2 = -18$

$$v = \begin{cases} t^2 - 9 & 0 \le t < 3 \\ 6t - 18 & t \ge 3 \end{cases}$$

b.



i Distance $= \int_0^3 |t^2 - 9| \mathrm{d}t + \int_3^5 6t - 18 \,\mathrm{d}t = 18 + 12 = 30$ m

ii Displacement is –18 + 12 = –6 m, so the distance from initial position is 6 m

16. a. $\frac{\mathrm{d}V}{\mathrm{d}t} = 4\sin(0.25t-2)+4-3\sin(0.25t)-4 = 4\sin(0.25t-2)-3\sin(0.25t)$

b. $V = \int (4\sin(0.25t-2)-3\sin(0.25t))\mathrm{d}t$

$V = -16\cos(0.25t-2)+12\cos(0.25t)+c$

at $t=0$, $V=100$, hence $c = 100+16\cos(-2)-12 \approx 81.3$

$V = -16\cos(0.25t-2)+12\cos(0.25t)+81.3$

c. From the GDC, the minimum amount is 57.6 litres

d. Either:

$V(24) \approx 103.32$, so the difference ≈ 3.32 litres

Or:

$V = \int_0^{24} |4\sin(0.25t-2)+4| \mathrm{d}t - \int_0^{24} |3\sin(0.25t)+4| \mathrm{d}t$

$= 3.32199\ldots \approx 3.32$ litres

17. a. $\frac{\mathrm{d}a}{\mathrm{d}t} = kab^2 \Rightarrow 12 = k \times 4 \times 2^2 \Rightarrow k = \frac{3}{4}$

b. $\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{3}{4}a(1.2-0.2t)^2 \Rightarrow \int \frac{4}{a}\mathrm{d}a = \int 3(1.2-0.2t)^2 \mathrm{d}t$

$4\ln a = -\frac{1}{0.2}(1.2-0.2t)^3 + c$

Note: the integral on the right can be done by multiplying out the brackets (in which case it will have a different form) or by inspection or using the substitution $u = 1.2-0.2t$

$4\ln a = -\frac{1}{0.2}(1.2-0.2t)^3 + c \Rightarrow \ln a = -\frac{5}{4}(1.2-0.2t)^3 + \frac{c}{4}$

$\Rightarrow a = Ae^{-\frac{5}{4}(1.2-0.2t)^3}$

When $t=0$, $a=0.5$ so $0.5 = Ae^{-\frac{5}{4}(1.2)^3} \Rightarrow A = 0.5e^{2.16} \approx 4.34$

So $a = 4.34e^{-\frac{5}{4}(1.2-0.2t)^3}$

c. $a = 4.34$ mg per litre when $t = 6$.

18. a. $a = v\dfrac{dv}{dx}$

$$\frac{dv}{dx} = \frac{x\times\left(-0.2xe^{-0.1x^2}\right)-e^{-0.1x^2}}{x^2} = \frac{-e^{-0.1x^2}\left(0.2x^2+1\right)}{x^2}$$

$$a = \frac{e^{-0.1x^2}}{x}\times\frac{-e^{-0.1x^2}\left(0.2x^2+1\right)}{x^2} = \frac{-e^{-0.2x^2}\left(0.2x^2+1\right)}{x^3}$$

b. $\dfrac{dx}{dt} = \dfrac{e^{-0.1x^2}}{x} \Rightarrow \int xe^{0.1x^2}dx = \int 1dt$

Let $u = 0.1x^2 \Rightarrow \dfrac{du}{dx} = 0.2x$

$$\int xe^u \frac{du}{0.2x} = t+c \Rightarrow 5\int e^u du = t+c$$

$$\Rightarrow 5e^{0.1x^2} = t+c \Rightarrow 0.1x^2 = \ln\left(\frac{t+c}{5}\right) \Rightarrow x = \sqrt{10\ln\left(\frac{t+c}{5}\right)} \text{ (as } x>0)$$

$x = 0$ when $t = 0 \Rightarrow 0 = \sqrt{10\ln\left(\dfrac{c}{5}\right)} \Rightarrow c = 5$ (since $\ln 1 = 0$)

$$x = \sqrt{10\ln\left(\frac{t+5}{5}\right)}$$

Topic 5 HL Paper 2

1. a. $\det\begin{pmatrix}1-\lambda & 2\\ 1 & -\lambda\end{pmatrix} = 0 \Rightarrow (1-\lambda)(-\lambda)-2 = 0 \Rightarrow \lambda^2-\lambda-2 = 0$

$\Rightarrow (\lambda-2)(\lambda+1) = 0 \Rightarrow \lambda = -1, 2$

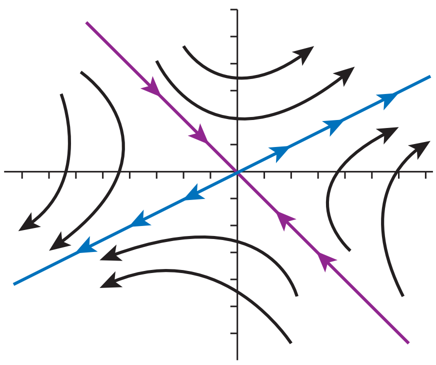
To find corresponding eigenvectors solve

$$\begin{pmatrix}1-\lambda & 2\\ 1 & -\lambda\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\lambda = -1 \Rightarrow \begin{pmatrix}2 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow 2x+2y = 0 \Rightarrow y = -x$$

$$\lambda = 2 \Rightarrow \begin{pmatrix}-1 & 2\\ 1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow -x+2y = 0 \Rightarrow y = \frac{1}{2}x$$

Possible eigenvectors are $\lambda = -1 \Rightarrow \begin{pmatrix}1\\ -1\end{pmatrix}$, $\lambda = 2 \Rightarrow \begin{pmatrix}2\\ 1\end{pmatrix}$



b. $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x = 3$ and $y = 4$ when $t = 0$

hence $\begin{matrix} 3 = A + 2B \\ 4 = -A + B \end{matrix}$

$B = \frac{7}{3}$ and $A = -\frac{5}{3}$

$x = -\frac{5}{3}e^{-t} + \frac{14}{3}e^{2t}$ and $y = \frac{5}{3}e^{-t} + \frac{7}{3}e^{2t}$

c. For large values of t $x \approx \frac{14}{3}e^{2t}$ and $y \approx \frac{7}{3}e^{2t}$, hence long term ratio is 2 : 1 (which is the same as the eigenvector corresponding to the largest eigenvalue).

2. **a.** $V = \pi\int_0^{\pi}(3\cos 2y + 4)^2 \, dy$

$= 202.32... \approx 202 \text{ cm}^3$

b. **i.** $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}, \frac{dV}{dt} = 2$

$\frac{dh}{dt} = \frac{2}{\pi(3\cos 2h + 4)^2}$

ii. $\frac{dh}{dt} = 0.039788... \approx 0.0398$

c. **i.** Let $\frac{dh}{dt} = \dot{h}$ and using the chain rule:

$\frac{d^2h}{dt^2} = \frac{d\dot{h}}{dt} = \frac{dh}{dt} \times \frac{d\dot{h}}{dh}$ (note this is similar to the expression in the formula books for acceleration $a = v\frac{dv}{ds} = \frac{ds}{dt} \times \frac{dv}{ds}$)

$\dot{h} = \frac{dh}{dt} = \frac{2}{\pi(3\cos 2h + 4)^2} = \frac{2}{\pi}(3\cos 2h + 4)^{-2}$

$\Rightarrow \frac{d\dot{h}}{dh} = \frac{-4}{\pi}(3\cos 2h + 4)^{-3} \times (-6\sin 2h) = \frac{24\sin 2h}{\pi(3\cos 2h + 4)^3}$

Hence $\frac{d^2h}{dt^2} = \frac{2}{\pi(3\cos 2h + 4)^2} \times \frac{24\sin 2h}{\pi(3\cos 2h + 4)^3} = \frac{48\sin 2h}{\pi^2(3\cos 2h + 4)^5}$

ii. Just require $\sin 2h = 0$ so $h = 0, \frac{\pi}{2}, \pi$

iii. $\frac{dh}{dt}$ is minimum at $h = 0, \pi$ as this is where the container is widest.

$\frac{dh}{dt}$ is maximum at $h = \frac{\pi}{2}$ as this is where the container is narrowest.

Topic 5 HL Paper 3

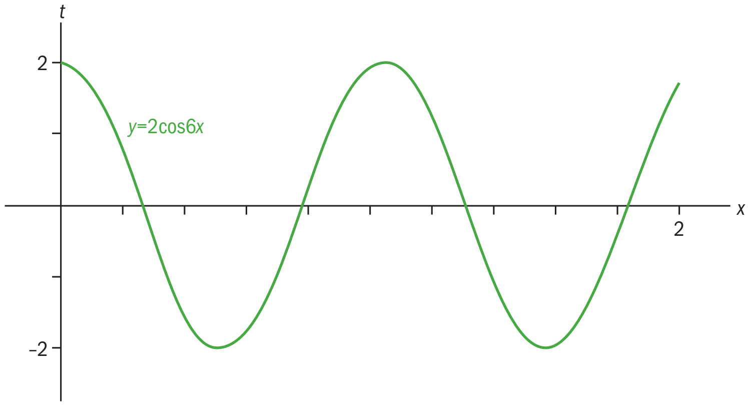
1. a. $x = A\sin 6t + B\cos 6t \Rightarrow \dot{x} = 6A\cos 6t - 6B\sin 6t$

$\Rightarrow \ddot{x} = -36A\sin 6t - 36B\cos 6t = -36(A\sin 6t + B\cos 6t) = -36x$

b. When $t = 0$, $x = 2 \Rightarrow 2 = B$, $\dot{x} = 0 \Rightarrow 6A = 0 \Rightarrow A = 0$

$x = 2\cos 6t$

c. i.



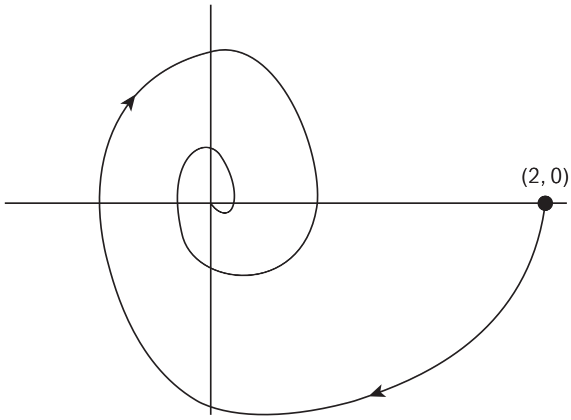
ii. The object oscillates about the equilibrium position with a constant amplitude of 2.

d. $\dot{x} = y,\ \dot{y} = -36x - 5y$

e. i. $\det\begin{pmatrix} 0-\lambda & 1 \\ -36 & -5-\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 + 5\lambda + 36 = 0$

$\lambda = \dfrac{-5 \pm \sqrt{-119}}{2} \approx -2.5 \pm 5.45\mathrm{i}$

ii. At $t = 0$, $\dot{y} = -72$ so the object is spiralling clockwise. Because the real part of the eigenvalue is negative it is spiralling inwards.



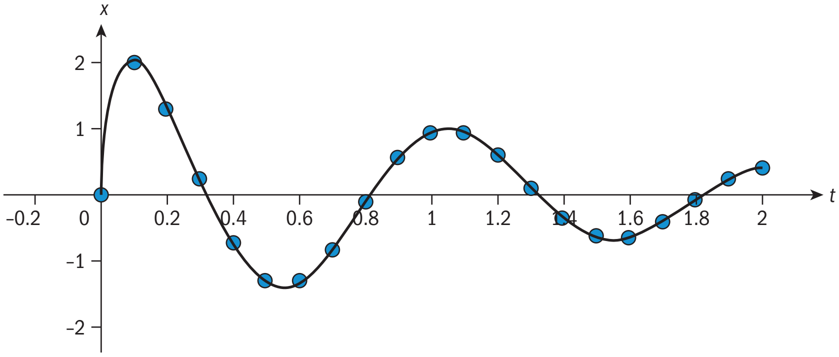
iii. This means the object is oscillating about O, with decreasing amplitude.

f. $\dot{x} = y,\ \dot{y} = -5y - 36x + 2\cos(t)$

g. $t_{n+1} = t_n + 0.1,\ x_{n+1} = x_n + 0.1y_n,\ y_{n+1} = y_n + 0.1(-5y_n - 36x_n + 2\cos(t_n))$

When $t = 1$, $x \approx 0.953$

h.



IB PREPARED MAI

ANSWERS TO PRACTICE EXAM PAPERS

Here are the answers to the practice exam papers from IB Prepared Mathematics: Applications and Interpretations.

For direct access, click on the paper below.

Awarding of marks

- 'M' indicates a mark for method. It might contain some incorrect values, but so long as the method is clear and correct this can be awarded.
- 'A' indicates a mark for the answer and is awarded if the correct answer is seen. Numerical answers should be exact or given to 3 significant figures unless the question specifies otherwise.
- 'R' indicates a mark for reasoning. This will often be in a question that contains the command terms 'explain' or 'justify'.
- A mark in brackets is for intermediate working that does not need to be seen. It is called an implied mark because a subsequent correct answer implies this line must have been completed even if it was not written down.

SL PRACTICE PAPER 1: MARKSCHEME

1. **a.** $k = 1 - 0.2 - 0.3 - 0.4 = 0.1$ [M1A1]

[2 marks]

b. $\mathrm{E}(X) = 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.4 + 4 \times 0.1 = 2.4$ [M1A1]

[2 marks]

[TOTAL 4 marks]

2. **a.** **i.** $228 - 150 = 78$ million km [M1]

(Note: M1 is for consideration of sum or difference and can be awarded in either part.)

7.8×10^6 km [A1]

ii. $150 + 228 = 378$ million km

3.78×10^6 km [A1]

[3 marks]

b. $\frac{100}{365} \times 2\pi \times 150 = 258.21\ldots \approx 258$ million kilometres M1A1

[2 marks]

[TOTAL 5 marks]

3.

Resort	A	B	C	D	E
Popularity rank	5	4	3	2	1
Hours of sunshine rank	4	5	2	3	1

[M1A1]

$r_s = 0.8$ [A2]

[TOTAL marks 4]

4. Radius of base = 0.5 m [A1]

Height = $\sqrt{1.2^2 - 0.5^2}$ [M1]

= 1.09087… m [A1]

Volume = $\frac{1}{3}\pi \times 0.5^2 \times 1.09087\ldots = 0.286$ m^3 (0.285589…) [M1A1]

[TOTAL 5 marks]

5. **a.** $P = \int 0.6x^2 - 3.6x + 5\,\mathrm{d}x$ [M1]

$P = 0.2x^3 - 1.8x^2 + 5x(+c)$ [A2]

(Note: A1 for one error, A0 for 2 or more errors.)

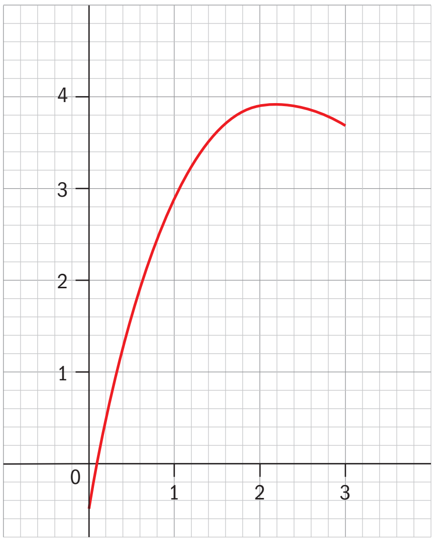
When $x = 0$, $P = -0.5$ [A1]

$c = -0.5$ [A1]

$P = 0.2x^3 - 1.8x + 5x - 0.5$

[5 marks]

b. i.



$x = 2.18350...$ [(M1)]

≈ 2180 kg [A1]

ii. 0 [A1]

[3 marks]

[TOTAL 8 marks]

6. a. i. $a = \dfrac{6.3 - 1.9}{2} = 2.2$ [(M1)A1]

ii. $b = \dfrac{360}{12} = 30$ [(M1)A1]

iii. $c = \dfrac{6.3 + 1.9}{2} = 4.1$ [A1]

[5 marks]

b. $2.2\cos(30t) + 4.1 = 4$ [(M1)]

$t = 3.0868...,\ 8.91315...$ [A1]

Percentage $= \dfrac{8.9131... - 3.0868...}{12} \times 100 = 48.552... \approx 48.6\%$ [A1]

[3 marks]

[TOTAL 8 marks]

7. H_0: Hearing ability and time listening to loud music are independent.

H_1: Hearing ability and time listening to loud music are not independent.

[A1]

p-value $= 0.39310... \approx 0.393$ [M1A1]

$0.393 > 0.05$ and so insufficient evidence to reject H_0 that hearing ability and time listening to loud music are independent. [M1R1]

[TOTAL 5 marks]

8. **a.** $V = kh^3$ [M1]

$225 = k \times 10^3 \Rightarrow k = 0.225$ [A1]

$V = 0.225h^3$ [A1]

[3 marks]

b. $C = n\sqrt{V}$ [M1]

$12 = \sqrt{225}n \Rightarrow n = 0.8$ [A1]

$C = 0.8\sqrt{V} = 0.8\sqrt{0.225h^3}$ [(M1)]

$C = 0.8\sqrt{0.225 \times 20^3} = 33.9411...$

$C \approx$ €33.94 [A1]

[4 marks]

[TOTAL 7 marks]

9. **a.** The plan view consists of 10 isosceles triangles, with vertex angle equal to 36°. [M1A1]

EITHER

$x = \dfrac{10}{\sin 18} = 32.360... \approx 32.4\,\text{cm}$ [M1A1]

OR

Use of cosine rule:

$20^2 = x^2 + x^2 - 2x^2 \cos 36$ [M1]

$x = \sqrt{\dfrac{400}{2 - 2\cos 36}} = 32.360... \approx 32.4$ [A1]

[4 marks]

b. Area $= 10 \times \dfrac{1}{2} \times 32.360... \times 32.360... \sin 36 = 3077.68... \approx 3080\,\text{cm}^2$ [M1A1]

[2 marks]

[TOTAL 6 marks]

10. Recognition of an arithmetic series [M1]

Distance required to walk $= 0 + 3 + 6 + 9 + ...$ [(A1)]

(Correct first term and difference – could start at 3)

$S_{20} = \dfrac{20}{2}(2 \times 0 + 19 \times 3) = 570\ \text{m}$ [M1A1A1]

[TOTAL 5 marks]

11. $a + b + c = 18000$ [A1]

$0.06a + 0.04b + 0.08c = 1210$ [M1A1]

$a = b + 4000$ [A1]

$a = 6500,\ b = 2500,\ c = 9000$ [(M1)A1]

[TOTAL 6 marks]

12. a. i. $\dfrac{dy}{dx} = -x$ [A1]

ii. When $x = 1$ $\dfrac{dy}{dx} = -1$ [M1]

$y - 4 = -(x - 1)$

$y = -x + 5$ [A1]

(Note: Any form of the equation is acceptable for A1.)

[3 marks]

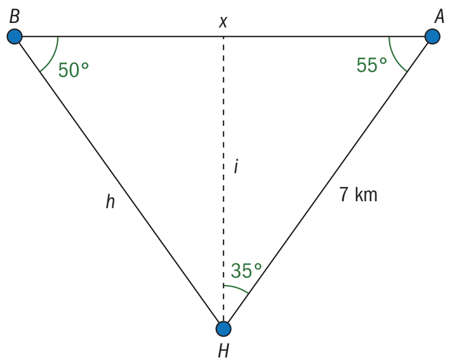
b. $y = -x + 5$ meets the x-axis at $x = 5$ [A1]

Length of ladder $\sqrt{(5-1)^2 + (4-0)^2} = \sqrt{32} \approx 5.66$ m [M1A1]

[3 marks]

[TOTAL 6 marks]

13.



$H\hat{B}A = 50°$ [A1]

$A\hat{H}B = 40 + 35 = 75°$ [M1A1]

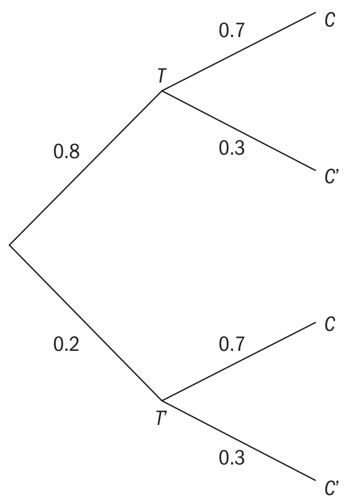
(or $H\hat{A}B = 55°$, $A\hat{H}B = 180 - 50 - 55 = 75°$)

$\dfrac{x}{\sin 75} = \dfrac{7}{\sin 50}$ [M1]

$x = 8.8264... \approx 8.83$ km [A1]

[TOTAL 5 marks]

14. a.



[A1]

[1 mark]

b. $1-0.7\times0.8=0.44$ [M1A1]

[2 marks]

c. $P(T'CT)+P(TC'C)$ [M1]

$0.2\times0.7\times0.8+0.8\times0.3\times0.7$ [A1]

$=0.28$ [A1]

[3 marks]

[TOTAL 6 marks]

SL PRACTICE PAPER 2 MARKSCHEME

1. a. i. 6.45 [A1]

ii. 6.2 [A1]

iii. 6.6 [A1]

[3 marks]

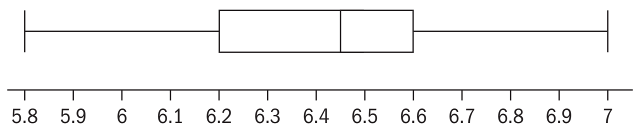
b. i. 0.4 [A1]

ii. $1.5 \times 0.4 = 0.6$ [M1]

$a = 5.6$, $b = 7.2$ [A1A1]

[4 marks]

c. i.



[M1A1]

(Note: M1 is for a box plot alongside an appropriate scale.)

ii. The distribution is symmetrical, or

The data is bunched around the centre, or

No outliers

(Award R1 for any of the above.) [R1]

[3 marks]

d. i. H_0: The mean height of the plants receiving nutrient A is equal to the mean height of the plants receiving nutrient B. [A1]

H_1: The mean height of the plants receiving nutrient A is less than the mean height of the plants receiving nutrient B. [A1]

(Note: Accept $\mu_A = \mu_B$ if both variables have been defined.)

ii. p-value = 0.00713 (0.0071328…) [M1A1]

$0.00713 < 0.01$ [R1]

iii. Therefore significant evidence that the mean height of the plants receiving nutrient A is less than the mean height of the plants receiving nutrient B. [A1]

(Note: Accept 'Reject H_0'.)

[6 marks]

[TOTAL 16 marks]

2. a. $400 \times 12 \times 20 = €\,96\,000$ [A1]

[1 mark]

b. $44\,451.64\ldots \approx €44\,451$ [(M1)A1]

[2 marks]

c. Loan $= 130\,000 - 20\,000 = 110\,000$ [A1]

Payment $= 788.074... \approx 788.07$ per month. [(M1)A1]

[3 marks]

d. In both cases Sumitra begins with €20 000

[Note: could begin with 0 and consider money in and out of the accounts]

Buying: $(20\,000) - 20\,000 - 20 \times 12 \times 788.07 + 130\,000 = -59\,136.80$ [A1][A1][A1]

[Note: A1 for $-20\,000$, A1 for $-20 \times 12 \times 788.07$ A1 for $+ 130\,000$]

Renting: $(20\,000) + 24451.64 - 96\,000 = -51\,548.36$ [A1][A1]

[Note: A1 for $+ 24\,451.64$, A1 for $-96\,000$]

It is better to rent. [R1]

The difference is €7588.44 [A1]

[Note: Accept any answer which rounds correctly to €7590.]

[7 marks]

[TOTAL 13 marks]

3. a. $P(X > 220) = 0.19568... \approx 0.196$ [(M1)A1]

[2 marks]

b. $P(X < a) = 0.15$ [(M1)]

$a = 153.72... \approx 154\text{ g}$ [(M1)A1]

[3 marks]

c. Large: $800 \times 0.19568... = 156.54... \approx 157$ [A1]

Small: $800 \times 0.15 = 120$ [A1]

Medium: $800 \times \text{P}(154 < X < 220) \approx 523$ [M1A1]

(or $800 - 157 - 120 = 523$)

[4 marks]

d. Distribution is $Y \sim \text{B}(5, 0.19568...)$ [M1A1]

$P(Y \geq 2) = 0.2538... \approx 0.254$ [M1A1]

(Note: Accept 0.255 from using $p = 0.196$.)

[4 marks]

[TOTAL 13 marks]

4. a. i. 1.2 [A1]

ii. Solve $3.4 = 1.2e^{b}$ [(M1)]

$b \approx 1.04$ [A1]

[3 marks]

b. $A = 1.2e^{1.04\times2} = 9.61\ (9.60536\ldots)\text{ cm}^2$ [A1]

[1 mark]

c. $A = 0.8 \times \pi \times 5^2 = 62.8318\ldots \text{cm}^2$ [(M1)A1]

$62.8418\ldots = 1.2e^{1.04t} \Rightarrow t = 3.8059\ldots \approx 3.81$ days [A1]

[3 marks]

d. $t = 3.81,\ A = 62.8318\ldots$

$t = 6.0,\ A = 25\pi \approx 78.5398\ldots$ [(M1)]

$62.8318\ldots = c + 3.81d$

$78.5398\ldots = c + 6d$ [M1A1]

$c = 35.5$ (accept 35.6) and $d = 7.17$ (accept 7.16) [(M1)A1]

[5 marks]

e. 7.17 cm^2 per day [A1]

[1 mark]

[TOTAL 13 marks]

5 a. $y = x - 1$ [A1A1]

[2 marks]

b. Midpoint is $(3, 4.5)$ [A1]

Gradient is $-\dfrac{2}{3}$ [(M1)A1]

$y - 4.5 = -\dfrac{2}{3}(x - 3)$ [M1]

$3y - 13.5 = -2x + 6$ [M1]

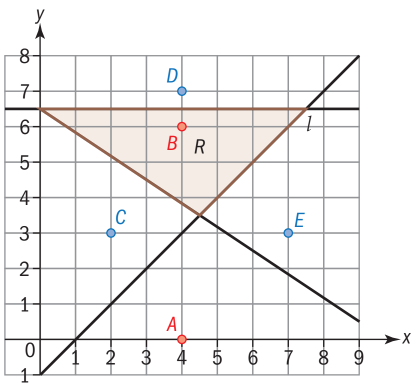
$2x + 3y = 19.5$ [AG]

[5 marks]

c. (4.5, 3.5) [(M1)A1]

[2 marks]

d. i.



[A1A1A1]

(Note: A1 for each of the perpendicular bisectors, A1 for correct region indicated.)

ii. Area = $\dfrac{1}{2} \times 7.5 \times 3 = 11.25\ \text{m}^2$. [M1A1]

[5 marks]

[TOTAL 14 marks]

6. a. i. 16% [A1]

ii. 19% [(M1)A1]

[3 marks]

b. f [A1]

Reason given: for example, because for f the poorest 50% of the people have 25% of the income and for g the poorest 50% of the people own 12.5% of the income. [R1]

[2 marks]

c. i. $p = 0.5 \Rightarrow w = 0.075$ or 7.5% [M1A1]

ii. Area under the curve is $\int_0^1 0.2p^3 + 0.6p^2 - 0.2p \; \mathrm{d}p$ [(M1)(A1)]

$= 0.15$ [A1]

$G = 1 - 2 \times 0.15 = 0.7$ [A1]

[6 marks]

[TOTAL 11 marks]

HL PRACTICE PAPER 1 MARKSCHEME

1. Radius of base = 0.5 m [A1]

Height = $\sqrt{1.2^2 - 0.5^2}$ [M1]

= 1.09087… m [A1]

Volume = $\frac{1}{3}\pi \times 0.5^2 \times 1.09087... = 0.286\ \text{m}^3$ (0.285589…) [M1A1]

[TOTAL 5 marks]

2. H_0: Hearing ability and time listening to loud music are independent.

H_1: Hearing ability and time listening to loud music are not independent.

[A1]

p-value $= 0.39310... \approx 0.393$ [M1A1]

0.393 > 0.05 and so insufficient evidence to reject H_0 that hearing ability and time listening to loud music are independent. [M1R1]

[TOTAL 5 marks]

3. Recognition of an arithmetic series [M1]

Distance required to walk = $0+3+6+9+...$ [(A1)]

(Correct first term and difference – could start at 3)

$S_{20} = \frac{20}{2}(2\times 0 + 19\times 3) = 570\,\text{m}$ [M1A1A1]

[TOTAL 5 marks]

4. $a+b+c = 18000$ [A1]

$0.06a + 0.04b + 0.08c = 1210$ [M1A1]

$a = b + 4000$ [A1]

$a = 6500,\ b = 2500,\ c = 9000$ [(M1)A1]

[TOTAL 6 marks]

5. a. i. $\frac{dy}{dx} = -x$ [A1]

ii. When $x = 1$ $\frac{dy}{dx} = -1$ [M1]

$y - 4 = -(x-1)$

$y = -x + 5$ [A1]

(Note: Any form of the equation is acceptable for A1.)

[3 marks]

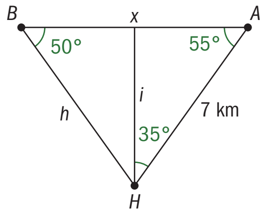
b. $y = -x + 5$ meets the x-axis at $x = 5$ [A1]

Length of ladder $\sqrt{(5-1)^2 + (4-0)^2} = \sqrt{32} \approx 5.66$ m [M1A1]

[3 marks]

[TOTAL 6 marks]

6.



$H\hat{B}A = 50°$ [A1]

$A\hat{H}B = 40 + 35 = 75°$ [M1A1]

(or $H\hat{A}B = 55°$, $A\hat{H}B = 180 - 50 - 55 = 75°$)

$$\frac{x}{\sin 75} = \frac{7}{\sin 50}$$ [M1]

$x = 8.8264... \approx 8.83$ km [A1]

[TOTAL 5 marks]

7. a. $\sqrt{4+4+1} = 3 \text{ km h}^{-1}$ [M1A1]

[2 marks]

b. $\boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ [A1]

[1 mark]

c. $\begin{pmatrix} 6 \\ 4 \\ 4 \end{pmatrix}$ A1

[1 mark]

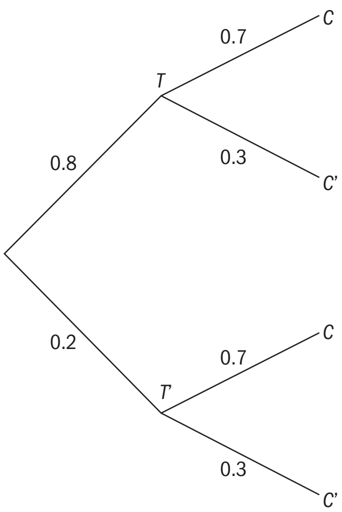
d. $\begin{pmatrix} 6 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$ [M1]

Distance = $\sqrt{4^2 + 2^2 + 4^2} = \sqrt{36} = 6$ km M1A1

[3 marks]

[TOTAL 7 marks]

8. a.



[A1A1]

[2 marks]

b. $1-0.7\times0.8=0.44$ [M1A1]

[2 marks]

c. $P(T'CT)+P(TC'C)$ [M1]

$0.2\times0.7\times0.8+0.8\times0.3\times0.7$ [A1]

$=0.28$ [A1]

[3 marks]

[TOTAL 7 marks]

9. a. $r=-0.985$ [A1]

[1 mark]

b. $\ln R=(-0.26699\ldots)\ln(m)-0.389699\ldots$ [A1A1]

[2 marks]

c. $R=e^{-0.2669\ldots\ln(m)-0.38969\ldots}$ [M1]

$R=e^{-0.38969\ldots}m^{-0.2669\ldots}$

$R\approx0.677m^{-0.267}$ [A1A1]

[3 marks]

d. $R=0.67726\ldots\times3850^{-0.2669\ldots}=0.07472\ldots\approx0.075$ [A1]

[1 mark]

e. Percentage error is $\dfrac{0.075-0.072}{0.072}\times100=4.166\ldots\approx4.2\%$ [A1]

[1 mark]

[TOTAL 8 marks]

10. a. i. $\dfrac{1}{(a-3)(a+2)+4}\begin{pmatrix}a+2 & 4\\ -1 & a-3\end{pmatrix}\left(=\dfrac{1}{a^2-a-2}\begin{pmatrix}a+2 & 4\\ -1 & a-3\end{pmatrix}\right)$ [M1A1]

ii. $a^2-a-2=0$ [M1]

$(a-2)(a+1)=0$

$a=2,-1$ [A1]

[4 marks]

b. $\mathbf{A}^{-1}=-\dfrac{1}{2}\begin{pmatrix}3 & 4\\ -1 & -2\end{pmatrix}\left(=\dfrac{1}{2}\begin{pmatrix}-3 & -4\\ 1 & 2\end{pmatrix}\right)$ [A1]

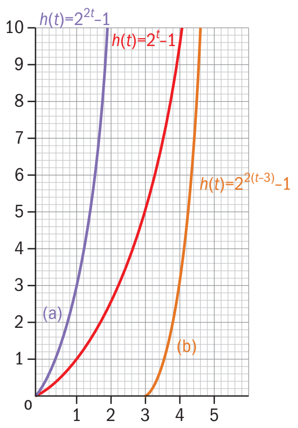
$\mathbf{S}=\dfrac{1}{2}\begin{pmatrix}-3 & -4\\ 1 & 2\end{pmatrix}\begin{pmatrix}2 & -4 & 6\\ -10 & 8 & -3\end{pmatrix}$ [M1]

$=\begin{pmatrix}17 & -10 & -3\\ -9 & 6 & 0\end{pmatrix}$ [A1]

[3 marks]

[TOTAL 7 marks]

11. a. i.



(curve concave up curve, always above h) [A1]

(curve passes through (0, 0) and (1, 3)) [A1]

ii. $h(t)=2^{2t}-1$ [M1A1]

[4 marks]

b. i. Translation of their curve 3 units to the right [A1]

ii. $h(t)=2^{2(t-3)}-1,\ t\geq 3$ [M1A1]

(Note: Award [M1A0] for $h(t)=2^{2(t+3)}-1$ or $h(t)=2^{2t-3}-1$)

[3 marks]

[TOTAL 7 marks]

12. Vertices of odd degree are A, B, F, C [M1]

Pair them as AB, FC

Pairs	Minimum time to travel between vertices
AB, FC	6, 14
AF, BC	12, 7
AC, BF	12, 12

[M1A1]

Walk should begin at either F or C and finish at the other vertex. [A1]

Length of walk is 75 + 6 = 81 minutes. [A1]

[TOTAL 5 marks]

13. a. $\begin{pmatrix} 0.4 & 0.8 \\ 0.6 & 0.2 \end{pmatrix}$ [M1A1]

b. $\begin{pmatrix} 0.4 & 0.8 \\ 0.6 & 0.2 \end{pmatrix}^3 = \begin{pmatrix} 0.544 & 0.608 \\ 0.456 & 0.392 \end{pmatrix}$ [M1]

Probability is 0.392 [A1]

c. $\begin{pmatrix} 0.4 & 0.8 \\ 0.6 & 0.2 \end{pmatrix}^4 \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.56704 \\ 0.43296 \end{pmatrix}$ [M1]

Probability is 0.567 [A1]

[TOTAL 6 marks]

14. a. i. $\tan\theta = \sqrt{3} \Rightarrow \theta = 60^0$ [M1A1]

From the formula book the matrix is $\begin{pmatrix} \cos(120°) & \sin(120°) \\ \sin(120°) & -\cos(120°) \end{pmatrix}$ [A1]

$\left(= \begin{pmatrix} -0.5 & 0.866... \\ 0.866... & 0.5 \end{pmatrix} \right)$

ii. $\begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix} \left(= \begin{pmatrix} 0.5 & -0.866... \\ 0.866... & 0.5 \end{pmatrix} \right)$ [A1]

[4 marks]

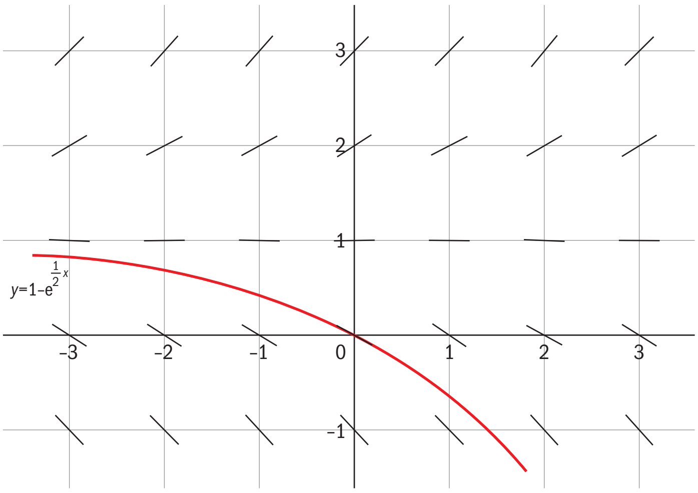
b. $\mathbf{T} = \begin{pmatrix} 0.5 & -0.866... \\ 0.866... & 0.5 \end{pmatrix} \begin{pmatrix} -0.5 & 0.866... \\ 0.866... & 0.5 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ [M1A1]

c. A reflection in the y-axis [(M1)A1]

[4 marks]

[TOTAL 8 marks]

15. a.



[A1]

[1 mark]

b. $\dfrac{dy}{dx} = \dfrac{y-1}{2} \Rightarrow \int \dfrac{2}{y-1}\,dy = \int 1\,dx$ [M1]

$2\ln(y-1) = x + c$ [A1]

$y = Ae^{\frac{1}{2}x} + 1$ [A1]

Passes through (0, 0), so $0 = A + 1 \Rightarrow A = -1$ [M1]

$y = 1 - e^{\frac{1}{2}x}$ [A1]

[5 marks]

[TOTAL 6 marks]

16. **a.** Let T be the total number of points.

i. $E(T) = 3 \times 3.2 + 2 \times 4.5 = 18.6$ [M1A1]

ii. Variance of a Poisson is equal to the mean so sight of 3.2 and 4.5 [M1]

$\text{Var}(T) = 9 \times 3.2 + 4 \times 4.5 = 46.8$ [M1A1]

[5 marks]

b. The sample is large enough for the central limit theorem to apply [R1]

[1 mark]

c. The distribution of the mean score in a season is

$\bar{X} \sim \text{N}\left(18.6, \frac{46.8}{40}\right)$ [M1A1]

(Note: M1 is for dividing the variance by 40.)

$P(\bar{X} > 20) = 0.0977808... \approx 0.0978$ [A1]

[3 marks]

[TOTAL 9 marks]

17. **a.** $\frac{dP}{dt} = \frac{ke^{-t}}{(1+e^{-t})^2}$ [M1A1]

[2 marks]

b. $\frac{dP}{dt} = \frac{ke^{-t}}{(1+e^{-t})^2}$

$\frac{d^2P}{dt^2} = \frac{-ke^{-t}(1+e^{-t})^2 - ke^{-t} \times 2(-e^{-t})(1+e^{-t})}{(1+e^{-t})^4}$ [M1A1]

$\frac{d^2P}{dt^2} = \frac{ke^{-t}(-1-e^{-t}+2e^{-t})}{(1+e^{-t})^3}\left(= \frac{ke^{-t}(e^{-t}-1)}{(1+e^{-t})^3}\right)$ [M1]

(Note: M1 is for cancelling $(1+e^{-t})$ or discounting it as a root after setting the expression equal to 0.)

$\frac{ke^{-t}(e^{-t}-1)}{(1+e^{-t})^3} = 0 \Rightarrow e^{-t} = 1 (\Rightarrow t = 0)$

[M1A1]

$P = \frac{k}{1+1} = \frac{k}{2}$ [M1]

[6 marks]

[TOTAL 8 marks]

HL PRACTICE PAPER 2 MARKSCHEME

1. **a.** **i.** 1.2 [A1]

ii. Solve $3.4 = 1.2e^{b}$ [(M1)]

$b \approx 1.04$ [A1]

[3 marks]

b. $A = 1.2e^{1.04\times 2} = 9.61\ (9.60536\ldots)\ \text{cm}^2$ [A1]

[1 mark]

c. $A = 0.8 \times \pi \times 5^2 = 62.8318\ldots\ \text{cm}^2$ [(M1)A1]

$62.8418\ldots = 1.2e^{1.04t} \Rightarrow t = 3.8059\ldots \approx 3.81\ \text{days}$ [A1]

[3 marks]

d. $t = 3.81,\ A = 62.8318\ldots$

$t = 6.0,\ A = 25\pi \approx 78.5398\ldots$ [(M1)]

$62.8318\ldots = c + 3.81d$

$78.5398\ldots = c + 6d$ [M1A1]

$c = 35.5$ (accept 35.6) and $d = 7.17$ (accept 7.16) [(M1)A1]

[5 marks]

e. 7.17 cm^2 per day [A1]

[1 mark]

[TOTAL 13 marks]

2. **a.** $y = x - 1$ [A1A1]

[2 marks]

b. Midpoint is $(3, 4.5)$ [A1]

Gradient is $-\dfrac{2}{3}$ [(M1)A1]

$y - 4.5 = -\dfrac{2}{3}(x - 3)$ [M1]

$3y - 13.5 = -2x + 6$ [M1]

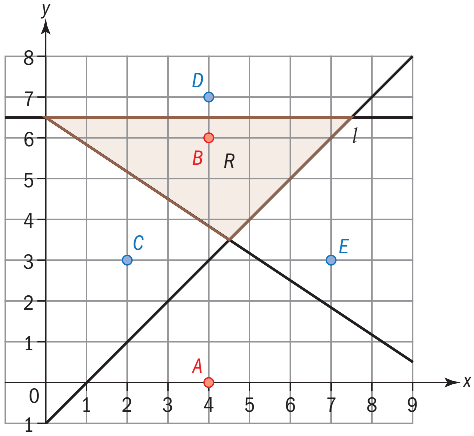
$2x + 3y = 19.5$ [AG]

[5 marks]

c. (4.5, 3.5) [(M1)A1]

[2 marks]

d. **i.**



[A1A1A1]

(Note: A1 for each of the perpendicular bisectors, A1 for correct region indicated.)

ii. Area = $\frac{1}{2}\times 7.5\times 3 = 11.25$ m^2. [M1A1]

[5 marks]

[TOTAL 14 marks]

3. **a.** **i.** 16% [A1]

ii. 19% [(M1)A1]

[3 marks]

b. f [A1]

Reason given: for example, because for f the poorest 50% of the people have 25% of the income and for g the poorest 50% of the people own 12.5% of the income. [R1]

[2 marks]

c. **i.** (0, 0) and (1, 1) [A1]

(0.5, 0.28) and (0.8, 0.65) [A1]

ii. Curve is $y = 0.19166...x^3 + 0.5925x^2 + 0.215833...x$ [M1A1]

Area under the curve is

$\int_0^1 \left(0.19166...x^3 + 0.5925x^2 + 0.2158...x\right) \mathrm{d}x$ [(M1)(A1)]

$= 0.3533...$ [A1]

$G = 1 - 2\times 0.3533... = 0.293$ [A1]

[7 marks]

[TOTAL 12 marks]

4. **a.** $\det\begin{pmatrix} 2-\lambda & -2 \\ 1 & 5-\lambda \end{pmatrix} = 0$ [M1]

$\Rightarrow (2-\lambda)(5-\lambda)+2=0$ [(A1)]

$\lambda^2 - 7\lambda + 12 = 0$

eigenvalues $\lambda = 3, 4$ [A1A1]

$\lambda = 3 \Rightarrow$

$$\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ [M1]

$x = -2y$, eigenvector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ [A1]

$\lambda = 4 \Rightarrow$

$x = -y$, eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ [A1]

[7 marks]

b. $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + Be^{4t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ [M1A1]

At $t = 0$, $2 = -2A - B \Rightarrow 4 = A + B$ [M1A1]

$A = -6,\ B = 10$ [A1A1]

$$\begin{pmatrix} x \\ y \end{pmatrix} = -6e^{3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 10e^{4t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\left(= e^{3t}\begin{pmatrix} 12 \\ -6 \end{pmatrix} + e^{4t}\begin{pmatrix} -10 \\ 10 \end{pmatrix}\right)$$

[6 marks]

[TOTAL 13 marks]

5. a. $x = \sqrt{5y},\ (0 \le y \le 20)$ [M1A1]

$V = \int_0^h \pi(5y)\,dy$ [M1A1]

$= \pi\left[2.5y^2\right]_0^h$ [A1]

$= 2.5\pi h^2 \text{ cm}^3$ [A1]

[6 marks]

b. Area $= \pi x^2 \text{ cm}^2$ [M1]

Rate $= 0.02 \times 5\pi h = 0.1\pi h \text{ cm}^3 \text{ s}^{-1}$ [M1A1]

[3 marks]

c. i. $5 = 0.1\pi h$ [M1]

$h = \dfrac{50}{\pi} = 15.915... \approx 15.9 \text{ cm}$ [A1]

[2 marks]

ii. $V = 2.5\pi \times (15.915...)^2 = 1989.4... \approx 1990 \text{ cm}^3$ [M1A1]

[2 marks]

[TOTAL 13 marks]

6. a. G_1 1 triangle

G_2 2 triangles

G_3 4 triangles [A1A1]

[Note: A1 for both **i.** and **ii.** and A1 for **iii.**]

[2 marks]

b. i. $\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ [(M1)A1]

ii. $\mathbf{M}^3 = \begin{pmatrix} 4 & 5 & 5 & 5 \\ 5 & 2 & 5 & 2 \\ 5 & 5 & 4 & 5 \\ 5 & 2 & 5 & 2 \end{pmatrix}$ [M1]

Number of walks = 4 [A1]

iii. Add up the entries in the leading diagonal of $\mathbf{M}^3$ (to give the number of cycles of length 3) [A1]

Divide the total by 6 because each triangle is counted 6 times. [A1]

From each vertex the triangle can be traversed in two directions [R1]

A cycle can begin from each of the three vertices in the triangle. [R1]

iv. Total number of cycles of length 3 $= 4+2+4+2=12$

$\frac{12}{6}=2$ [A1]

The method is verified as this is equal to the value obtained by inspection. **[9 marks]**

[TOTAL 11 marks]

7. a. $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 12 \\ 0 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 5 \\ 0 \\ 1.8 \end{pmatrix}$ [M1A1A1]

[3 marks]

b. $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 21.6 \\ 0 \\ -60 \end{pmatrix}$ [M1A1]

[2 marks]

c. $\left|\begin{pmatrix} 21.6 \\ 0 \\ -60 \end{pmatrix}\right| = \sqrt{21.6^2+60^2} = 63.8\text{ m}^2$ [M1M1A1]

[3 marks]

d. $k\sqrt{8^2+4^2+1^2} = k\sqrt{81} = 900$ [M1]

$\Rightarrow k = 100$ [A1]

[2 marks]

e. i. Vector perpendicular to the roof is $\begin{pmatrix} 21.6 \\ 0 \\ -60 \end{pmatrix}$ [A1]

ii. Using $\frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{b}|}$, component of intensity along this vector is

$\frac{100}{63.8}\begin{pmatrix} 21.6 \\ 0 \\ -60 \end{pmatrix}\bullet\begin{pmatrix} 8 \\ 4 \\ -1 \end{pmatrix}$ [M1A1]

$= \frac{100\times 232.8}{63.8\ldots} \approx 365$ watts per m^2 [M1A1]

[5 marks]

f. $365.06... \times 63.8... \times 0.2$ [M1]

≈ 4656 watts [A1]

[2 marks]

[TOTAL 17 marks]

8. a. $0.158\,655... \approx 0.1587$ [(M1)A1]

[2 marks]

b. i. Let the number that fail be Y.

$Y \sim B(5, 0.1587)$ [(M1)A1]

$P(Y \geq 2) = 0.181\,029... \approx 0.181$ [(M1)A1]

ii. $5000 \times 10 \times 0.181\,029... = 9051.5... \approx 9050$ [A1]

[5 marks]

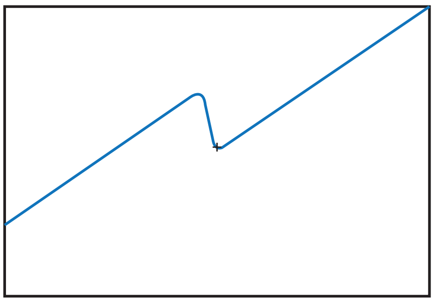
c. i. $\bar{X} \sim N\left(1000, \frac{50^2}{5}\right) = N(1000, 500)$ [M1A1A1]

ii. $P(\bar{X} < a) = 0.181\,029... \Rightarrow a = 979.619... \approx 980$ g [M1A1]

[5 marks]

d. i. $C = 5000 \times P(\bar{X} < 980) + 10\,000 \times \frac{1}{1000} \times \mu$ [M1A1A1]

$\left(= 5000 normcdf\left(\mu, \frac{50}{\sqrt{5}}, 980\right) + 10\mu \right)$



$\mu = 1027$ g [A1]

ii. The expected cost is less if $\mu < 536$ but they would not sell many 1 kg bags if the weight was so much less. [R1]

[5 marks]

[TOTAL 17 marks]

HL PRACTICE PAPER 3 MARKSCHEME

1. a. i. 5 [A1]

ii. Let p be the probability George beats Albert.

H_0: $p = 0.5$, H_1: $p > 0.5$ [A1]

Let X be the number of times that George wins

Under H_0, $X \sim B(8, 0.5)$ [(M1)]

$P(X \geq 5) = 0.363\,281... \approx 0.363$ [(M1)A1]

$0.363 > 0.05$ so insufficient evidence to reject H_0 [R1A1]

[7 marks]

b. Let μ_D = (mean time for Albert) – (mean time for George)

H_0: $\mu_D = 0$, H_1: $\mu_D > 0$ [A1A1]

(Note: The hypotheses can be expressed in words, but needs to be clear which way around the subtraction is made.)

Difference	1.1	1.4	−0.1	1.6	1.4	−3.3	−0.1	0.1

[M1A1]

p-value = 0.328 962… ≈ 0.329 > 0.05 [A1R1]

Insufficient evidence to reject H_0 that there is no difference in their average times. [A1]

[7 marks]

c. i. Any valid reason, for example data is discrete, or data is bounded. [R1]

ii. The sample is sufficiently large for the **central limit theorem** to apply, and so we can assume that the distribution of the sample mean is normal. [R1]

iii. Let the mean position for George be μ_G and

the mean position for Albert be μ_A.

H_0: $\mu_G = \mu_A$, H_1: $\mu_G < \mu_A$ [A1A1]

Unbiased estimators for the population variances are

$\frac{35}{34} \times 2.12^2$ and $\frac{35}{34} \times 2.56^2$ [(M1)(A1)]

p-value = 0.045 064… < 0.05 [A1R1]

The result is significant so we accept the alternative hypothesis that George's average position is less than Albert's average position. [A1]

[9 marks]

d. The final test indicated that George was a better skier than Albert but it used information from the previous two seasons. In the eight most recent races there was no significant difference, which indicates that Albert has improved relative to George (or George has got worse). [R1R1]

[2 marks]

[TOTAL 25 marks]

2. a. i. $(\cos t+\mathrm{i}\sin t)^2=\cos^2 t-\sin^2 t+2\mathrm{i}\sin t\cos t$ [M1A1]

ii. $\cos 2t+\mathrm{i}\sin 2t \Rightarrow r=1,\ a=2$ [A1A1]

iii. Equating real parts: [M1]

$\cos 2t=\cos^2 t-\sin^2 t$

$=1-\sin^2 t-\sin^2 t$ [M1]

$=1-2\sin^2 t$ [AG]

[6 marks]

b. $\int_0^{2\pi}\sin 2t\sin t\,\mathrm{d}t=0$ [A1]

[1 mark]

c. i. $\sin^2 t=\frac{1}{2}(1-\cos 2t)$ [A1]

ii. $\int \sin^2 t\,\mathrm{d}t=\frac{1}{2}t-\frac{1}{4}\sin 2t(+c)$ [A1A1]

iii. $\int_0^{2\pi}\sin^2 t\,\mathrm{d}t=\frac{1}{2}\left[t-\frac{1}{2}\sin 2t\right]_0^{2\pi}$

$=\frac{1}{2}(2\pi-0)$ [M1A1]

$=\pi$ [AG]

[5 marks]

d. $\int_0^{2\pi}(a_1\sin t+a_2\sin 2t)\sin t\,\mathrm{d}t$ [A1]

$=a_1\int_0^{2\pi}\sin^2 t\,\mathrm{d}t+a_2\int_0^{2\pi}\sin 2t\sin t\,\mathrm{d}t$ [M1]

$=a_1\pi$ [AG]

[2 marks]

e. $p=-2.09$ [M1A1]

[2 marks]

f. i. $\int_0^{2\pi}f(t)\sin t\,\mathrm{d}t\approx\frac{1}{2}\times\frac{\pi}{4}(0+2(3.29+1.19-2.09+0-2.08+1.21+3.29)+0)$

[(M1)(A1)]

$=3.78.\ (3.777765)$ [A1]

ii. $\frac{3.777765}{\pi}=1.20\,(1.2025...)$ [M1A1]

[5 marks]

g. $\int_0^{2\pi}f(t)\sin(2t)\mathrm{d}t=\int_0^{2\pi}(a_1\sin t+a_2\sin 2t)\sin(2t)\,\mathrm{d}t$

$=\int_0^{2\pi}a_1\sin t\sin(2t)\,\mathrm{d}t+\int_0^{2\pi}a_2\sin^2(2t)\,\mathrm{d}t$ [M1]

$=0+\int_0^{2\pi}a_2\sin^2 2t\mathrm{d}t$ [A1]

$=\frac{a_2}{2}\int_0^{2\pi}1-\cos(4t)\,\mathrm{d}t$ [A1]

$=\frac{a_2}{2}\left[t-\frac{1}{4}\sin(4t)\right]_0^{2\pi}$ [A1]

$=\frac{a_2}{2}\times 2\pi=\pi a_2$ [A1]

[5 marks]

h. i. $\int_0^{2\pi} f(t)\sin(2t)\mathrm{d}t \approx 11.9 \ (11.9380...)$ [(M1)A1]

(Note: there must be some evidence of the use of the trapezium rule.)

ii. $a_2 = \dfrac{11.938...}{\pi} = 3.80$ [A1]

$f(t) \approx 1.2\sin t + 3.8\sin(2t)$ [A1]

[4 marks]

[TOTAL 30 marks]